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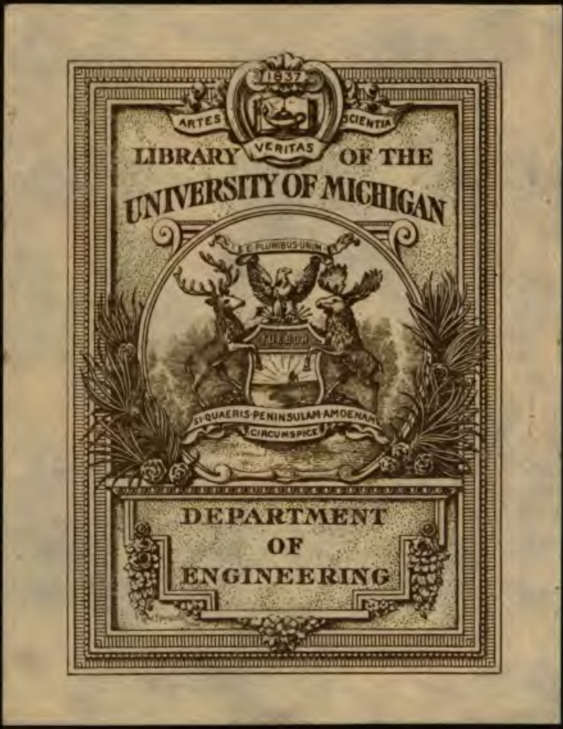
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ENGINEERING APPLICATIONS
OF
HIGHER MATHEMATICS

BY

V. KARAPETOFF

PART IV.
PROBLEMS ON MECHANICS OF MATERIALS

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FIRST THOUSAND

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PREFACE TO PART IV.

THE first part of this work which contains problems in machine design was published in 1912, and the favorable reception accorded it gave the author the courage to publish the rest of the problems which he had collected and used in his classes for some years. The second and third parts contain problems selected from various branches of hydraulics and thermodynamics respectively, in the solution of which it is necessary to use calculus and analytic geometry. The part now offered contains problems selected from various topics in mechanics of materials, and the last part contains problems on electrical engineering. A student or an engineer who wishes to review calculus or analytics, or to acquire facility in applications of higher mathematics to engineering problems, may thus select at first the part of the work which deals with problems with which he is most familiar, or in which he is particularly interested.

The reader will find the author's views on teaching mathematics to engineering students in the Preface to Part I and in the Dialogue following that preface. He is also referred to Part I for a list of reference works on mathematics and for an Appendix entitled "What a Senior in Engineering ought to know about Mathematics."

The author wishes to acknowledge gratefully the assistance of Mr. A. C. Stevens, M.E., instructor in Cornell University, who read the proofs and made some valuable suggestions for the text.

CORNELL UNIVERSITY, ITHACA, N. Y.

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PART IV.

PROBLEMS ON
MECHANICS OF MATERIALS.

ENGINEERING MATHEMATICS.

CHAPTER I.

STRENGTH OF SIMPLE HORIZONTAL BEAMS.

LET a straight horizontal beam of uniform cross section (Fig. 1) be freely supported on two knife-edges, and loaded with vertical forces or weights.

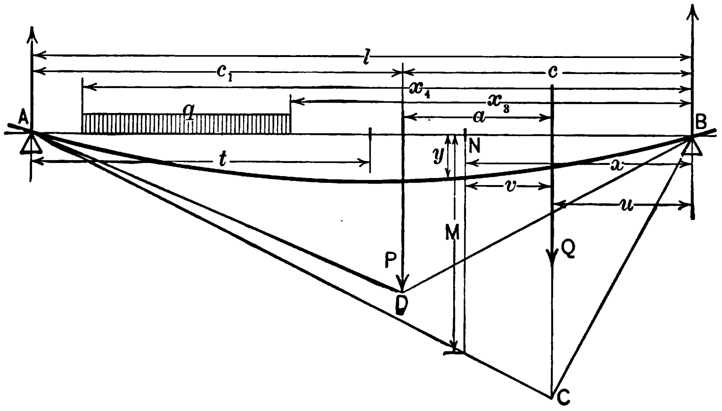

FIG. 1. — A loaded horizontal beam.

Some of these forces or loads may be concentrated, such as P and Q, others uniformly or non-uniformly distributed along the whole span or over a portion of it. If a load is distributed it may be given as q kg. per running meter of the span, where q may be either a constant quantity or a function of the distance x from one of the supports.

Under the influence of these forces the beam is deflected, and its longitudinal axis AB becomes a curve.

The cross sections of the beam which are originally perpendicular to this axis, and which lie in parallel planes before the forces are applied, are now deformed by the stresses. A fundamental assumption is made in the theory of the strength of materials that as long as the stresses in the beam do not exceed a certain "elastic" limit, the plane cross sections of the beam remain plane in their new distorted position (Fig. 2). The beam may be considered to consist of longitudinal fibers

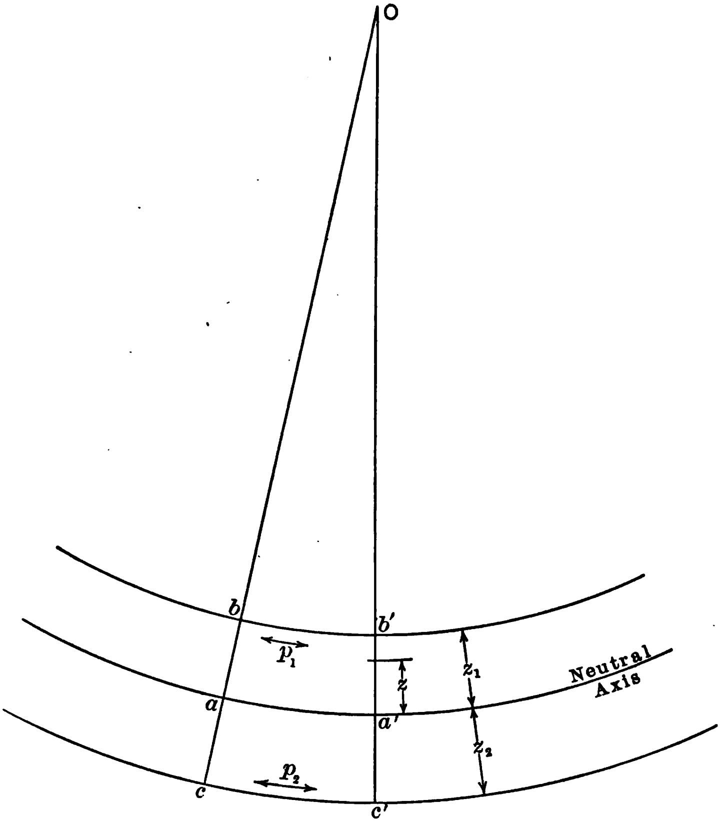

FIG. 2. — The axis and two plane cross sections of a loaded beam.

parallel to the axis, and the deformation of the beam causes a longitudinal compression along the fibers on the concave side of the beam (bb') and a longitudinal tension along the fibers on the convex side (cc'). There is a set of fibers, such as aa', which do not change in length during the deformation. They all lie before the deformation in a horizontal plane called the neutral plane. All the fibers above this plane are in a state of compression, those below the neutral plane are in a state of tension.

It will be seen from Fig. 2 that maximum longitudinal stresses take place in the extreme fibers bb' and cc', so that if the stresses in these fibers do not exceed a safe permissible limit, the whole beam is safe.* The maximum stress varies from cross section to cross section of the beam, depending upon the position of the applied forces. Since the action of these forces is to bend the beam, the internal stresses in a cross section are proportional to the bending moment of the applied forces with respect to that cross section. For instance, in Fig. 1, the bending moment M with respect to cross section N is

$$M = Bx - Qv,$$

where B is the force reaction of the right-hand support. In this equation force Q is taken with the minus sign because its moment is opposite to that of force B. Since the beam is in equilibrium, the numerical value of the moment of the forces to the left of the cross section N is also equal to M, for otherwise the beam would be turning under the influence of an unbalanced moment. We took the moment of the forces to the right of N because in this case its expression happens to be simpler.

---

* The beam must also be checked for the shearing stress near the neutral plane. This stress is not considered here, and is assumed to be small; otherwise, the maximum combined stress does not necessarily take place in the extreme fibers.

Let the distance from the neutral axis (Fig. 2) to the extreme compressed fiber be z_1 cm., and let the stress in this fiber be p_1 kg. per sq. cm. The relationship between this stress and the bending moment M is

$$M = p_1 I / z_1. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

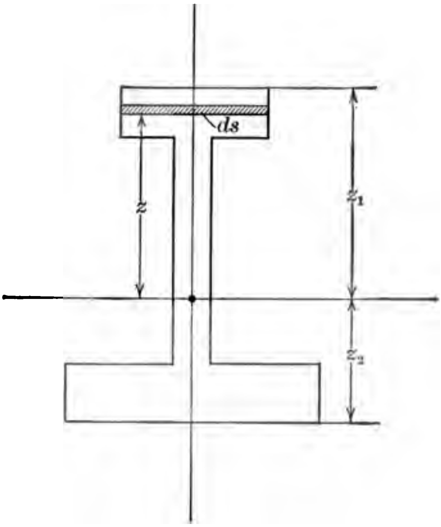

FIG. 3. — The cross section of a beam.

In this formula I stands for the moment of inertia of the cross section with respect to the horizontal axis which lies in its plane and passes through the center of gravity of the cross section.* Thus, referring to Fig. 3, which represents a cross section of the beam

$$I = \int_{-z_2}^{z_1} z^2\, ds, \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

---

* The student will find proof of this fundamental formula in any standard work on mechanics of materials.

where ds is an infinitesimal horizontal strip of the area, and z is the distance of this strip from the horizontal axis passing through the center of gravity.

For the maximum tensile stress in a cross section we can write, by analogy with Eq. (1), and using the notation in Fig. 2:

$$M = p_2 I / z_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Prob. 1. — Check the physical dimensions in formula (1). *Solution.* —

$$\text{kg.} \times \text{cm.} = \frac{\text{kg.}}{\text{cm.}^2} \times \text{cm.}^4/\text{cm.},$$

or, after simplification,

$$\text{kg.} \times \text{cm.} = \text{kg.} \times \text{cm.},$$

which is an identity.

Prob. 2. — Prove that for a rectangular cross section of width b and height h the moment of inertia $I = \frac{1}{12} bh^3$, and $I/z_1 = I/z_2 = \frac{1}{6} bh^2$.

Prob. 3. — Referring to the preceding problem, a rectangular beam is to be sawed out of a log of diameter D. Determine b and h in such a way as to get a beam that will carry the greatest possible load. *Hint.* $\frac{1}{6} bh^2 =$ max., and $b^2 + h^2 = D^2 =$ const.

Prob. 4. — A log is to be sawed into thin boards of equal width, and it is desired to utilize the material so as to have as little waste as possible. Compare per cent weight in the log utilized for boards with that which could be utilized for a beam of maximum strength as required in the preceding problem.

Prob. 5. — With a given factor of safety, how much more load can the original log carry than the strongest rectangular beam cut from it?

Prob. 6. — It is proved in mechanics of materials that the deflection of a beam under a given load is proportional to the moment of inertia I, and not to the ratio I/z_1. Modify Prob. 3 to read that the beam must be the stiffest possible and not the strongest possible.

Prob. 7. — Make a sketch showing the cross section of a log and of three beams that may be sawed out of it, namely, the beam of maximum cross section, the strongest beam, and the stiffest beam.

Prob. 8. — Make a sketch of the cross section of an I beam, denote its dimensions by suitable letters and deduce a general expression for its moment of inertia.

Prob. 9. — A rectangular horizontal plate (Fig. 4) is fastened in a wall and is loaded with a vertical force P at one edge. What is the weakest plane along which the fracture is most liable to occur? *Solution.* — Let the wavy line represent a possible plane of failure determined by an unknown angle α. Applying Eq. (1) we get (see Prob. 2) $Px = p \cdot \frac{1}{6} yh^2$, or $p = \text{const.} \times (x/y)$. Thus, the weakest plane, viz., the one for which the stress p is a maximum, is determined by the condition $x/y = \text{max.}$ Both x and y are functions of angle α, and we have $x = c \sin \alpha$, $y = c/\cos \alpha$, so that $\sin \alpha \cos \alpha = \text{max.}$ But $2 \sin \alpha \cos \alpha = \sin 2\alpha$, and its maximum value is reached when $\alpha = 45°$.*

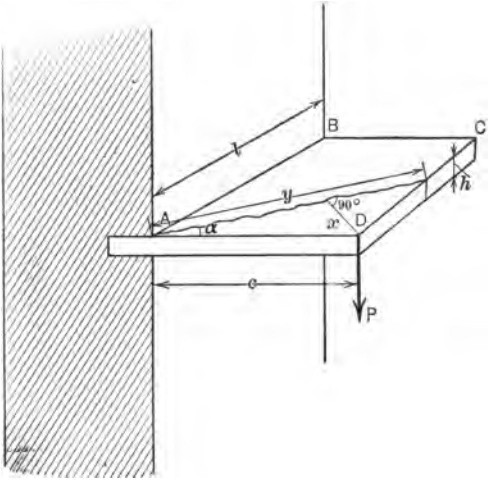

FIG. 4. — A horizontal plate loaded at one corner.

Prob. 10. — Find by direct differentiation the maximum value of $\sin \alpha \cos \alpha$ required in the preceding problem.

Prob. 11. — Referring to Prob. 9, how much stronger is the plate along planes corresponding to $\alpha = 40°$ and $\alpha = 50°$ than it is for the 45-degree plane?

Prob. 12. — Show that the answer to Prob. 9 is correct only when $l > 2c$ (Fig. 4). When $l = 2c$ the plate is theoretically just as liable to break diagonally along $\alpha = 45°$ as along the plane AB. When $l < 2c$ the plane AB is the weakest cross section.

---

* The conditions in this problem are approximately those which obtain between two meshing gears in the most unfavorable case when the pressure is transmitted at one corner of the tooth. The tooth is most liable to break along a 45-degree plane.

Prob. 13. — Prove that the plate (Fig. 4) is strongest along the diagonal plane AC.

Prob. 14. — Find a proof of formula (1) in some book, and give it in your own words.

Let the beam (Fig. 1) be loaded with the weight Q only, at a distance u from the right-hand support, and let it be required to find the dangerous cross section of the beam, that is, the one for which the bending moment due to the force Q is a maximum. Let this cross section be N, at an unknown distance x from the support B. According to the rule for the addition of parallel forces, the reactions of the supports are inversely as the distances from the load Q, and we have

$$\text{Reaction } A = \frac{u}{l}Q, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\text{Reaction } B = \frac{l-u}{l}Q, \quad . \quad . \quad . \quad (5)$$

the sum of the two reactions being equal and opposite to the force Q.

The bending moment with respect to point N is

$$M = \frac{u}{l}Q(l-x), \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and this expression holds true as long as no force is applied between A and N.* The moment M reaches its maximum for the smallest possible value of x, or in this case when $x = u$. Thus, with one concentrated load the maximum value of the bending moment takes place under the load, and is equal to

---

* In this chapter x generally denotes the distance from B to some point on the beam, such as N; u denotes the distance from B to the point of application of a force. In different problems either x or u or both may be variable.

$$M = \frac{x}{l}(l - x)\,Q. \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Eq. (6) shows that with a fixed load (u = const.) the bending moment varies along the beam according to a straight-line law. This is shown in Fig. 1 by the broken line ACB.

Prob. 15. — Show that with a moving load Q, the bending moment is a maximum when the load is in the center of the span.

Prob. 16. — A uniformly distributed load of q kg. per meter of span is applied to the beam as shown in Fig. 1, viz., between the abscissæ x_3 and x_4. Calculate the bending moment due to this load at a given point N. *Solution.* — The infinitesimal load at a distance u from B is $q\,du$, so that the moment at N due to this load is, according to Eq. (6), $dM = u\,(l - x)\,q\,du/l$. Integrating this expression between the limits $u = x_3$ and $u = x_4$, the desired moment due to the whole load is obtained.

Prob. 17. — The beam is loaded simultaneously with a concentrated weight Q at a distance u from support B (Fig. 1) and with a distributed load q per unit length between x_3 and x_4. Find the abscissa x for which the total bending moment is a maximum.

Prob. 18. — A load is non-uniformly distributed over a portion of the span, so that $q = f(u)$. Write down the general expression for M at point N, and apply it to a few simple cases, for example, (*a*) the load increases or decreases according to a straight-line law from the supports to the center of the span; (*b*) the load varies according to a parabolic law, either over the whole span, or from both supports towards the center; (*c*) the load varies according to an exponential law; (*d*) the load varies inversely as the nth power of the distance from one of the supports; (*e*) the load varies according to the sine law. In each case find point N for which the bending moment is a maximum.

Prob. 19. — Two unequal concentrated loads, P and Q (Fig. 1), are rigidly connected together at a constant distance a, and move along the span. Find the greatest bending moment which this system of loads exerts upon the span, and the dangerous cross section of the beam. Two connected moving loads may be due in practice to two pairs of wheels of a truck moving along a bridge. *Solution.* — The distribution of moment due to Q is represented by the line ACB, that due to P by the line ADB. The combined moment is obtained by adding the ordinates of these two curves. It will thus be seen that the maximum moment is under one of the loads, and not between or outside them. Assume first that max. M is under Q. Then for any position of Q, determined by an abscissa $u = x$, the total moment is $M = Qx\,(l - x)/l + P\,(x + a)(l - x)/l$, according to Eqs. (7) and (6). Take a derivative of M with respect to x and equate it to zero. The value of x from this equation will give the position of the dangerous cross section. Make similar calculations for weight P, and compare the values of max. M so obtained.

Prob. 20. — Referring to the preceding problem, prove that with P equal to Q, the combined load exerts its maximum moment when the distances from the center of the span to P and Q are equal to $\frac{1}{4}a$ and $\frac{3}{4}a$ respectively.

Prob. 21. — Extend Prob. 19 to the case of three unequal weights, P, Q, and R, at distances a and b respectively.

Prob. 22. — Extend Prob. 19 to the case when, in addition to P and Q, the beam is subjected to a uniform load q, for instance, its own weight. Note that in this case the maximum moment is not necessarily under one of the concentrated loads.

CHAPTER II.

THE ELASTIC CURVE OF A BEAM.

AN elastic curve is the curve which represents the shape of the neutral axis of a beam (Fig. 1) after flexure. It is necessary in some practical problems to know the equation of this curve; for instance,

(*a*) To calculate the maximum deflection of the beam;

(*b*) To determine the strength of a "constrained" beam, or a beam fixed at the ends.

(*c*) In the theory of continuous girders, or beams resting on more than two supports.

Let the abscissæ x of the elastic curve be measured from the right-hand support, and let the ordinates y be considered positive downwards. It is proved in the mechanics of materials that the differential equation of the elastic curve is

$$-\frac{d^2y}{dx^2}=\frac{M}{EI}, \quad \ldots \ldots \quad (1)$$

where M and I have the same meaning as in the preceding chapter, and E is the modulus of elasticity or a coefficient which characterizes the extensibility of the material. If the load on the beam is given, M is a known function of x, so that Eq. (1) may be integrated twice and the ordinate y of the elastic curve expressed as a function of x.

The minus sign is necessary in Eq. (1) if the moment M is to be considered positive when it bends the beam in such a way that the upper side is concave. An inspection of Fig. 1 will show that in this case d^2y/dx^2 is negative for the whole curve, and in order to make it positive the minus sign is necessary.

A few examples will make the use of Eq. (1) clear.

Case A. A beam is loaded with a concentrated weight P at the center of the span. The reaction of each support is $\frac{1}{2}P$, and the bending moment at a point determined by the abscissa x, when $x < \frac{1}{2}l$, is $M = \frac{1}{2}Px$. Substituting this value in Eq. (1) we get

$$-\frac{d^2y}{dx^2} = \frac{P}{2EI}x.$$

Integrating this expression once, we get

$$\frac{dy}{dx} = \frac{P}{2EI}(C_1 - \tfrac{1}{2}x^2), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where C_1 is the constant of integration. Integrating, again, gives

$$y = \frac{P}{2EI}(C_2 + C_1x - \tfrac{1}{6}x^3). \quad . \quad . \quad . \quad (3)$$

The constants C_1 and C_2 are determined from the following conditions: when

$$x = \tfrac{1}{2}l, \quad \frac{dy}{dx} = 0;$$

when $x = 0, \quad y = 0.$

Using the first condition in Eq. (2), we get

$$C_1 = \tfrac{1}{8}l^2;$$

the second condition used in Eq. (3) gives

$$C_2 = 0,$$

so that finally

$$y = \frac{P}{4EI}(\tfrac{1}{4}l^2x - \tfrac{1}{3}x^3). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The deflection is a maximum in the center of the span, and its value is

$$y_{\max} = \frac{Pl^3}{48\,EI}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Case B. Let now the weight P be applied at a distance c from the right-hand support. In this case, on account of the unsymmetrical load, it is necessary to consider parts of the beam to the left and to the right of the load, because it is not possible to say offhand on which side the deflection is a maximum. The reaction of the support B is equal to $P\ (l - c)/l$, so that by analogy with Eq. (6) in the preceding chapter, the moment at a distance x from B (for $x < c$) is $M = Px\ (l - c)/l$. Thus for a point on the right-hand side of the beam we have:

$$-\frac{d^2y}{dx^2} = \frac{P\,(l - c)}{EIl}\,x, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or, after integration,

$$\frac{dy}{dx} = \frac{P\,(l - c)}{EIl}\,(C - \tfrac{1}{2}x^2), \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and

$$y = \frac{P\,(l - c)}{EIl}\,(C' + Cx - \tfrac{1}{6}\,x^3), \quad . \quad . \quad . \quad (8)$$

where C and C' are constants of integration, to be determined from the terminal conditions of the problem; namely, when $x = 0$, $y = 0$, so that Eq. (8) gives $C' = 0$. The constant C must be determined from the condition that the deflection y is the same for the right-hand side of the beam as for the left-hand side, when $x = c$. Thus, it is necessary to write an equation corresponding to Eq. (8), for the left-hand side of the beam.

Were we to use the same origin B for the left-hand side of the beam this equation would be of a different form from Eq. (8), and the determination of the constants of integration would be somewhat tedious. In order to remedy this, we transfer the origin to A, and write directly, by analogy with Eqs. (7) and (8):

$$\frac{dy}{dt} = \frac{P(l - c_1)}{EIl}(\Gamma - \tfrac{1}{2}t^2), \quad . \quad . \quad . \quad . \quad (9)$$

and

$$y = \frac{P(l - c_1)}{EIl}(\Gamma t - \tfrac{1}{6}t^3). \quad . \quad . \quad . \quad (10)$$

In these equations t is the abscissa measured from the origin A, Γ is a constant of integration, and c_1 is the distance from the support A to the force P (Fig. 1), so that

$$c + c_1 = l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

At $x = c$, and $t = c_1$, the two elastic curves, represented by Eqs. (8) and (10), have a common ordinate and a common tangent. Or

$$\left.\begin{aligned} y_c &= y_{c_1}, \\ -\left(\frac{dy}{dx}\right)_c &= \left(\frac{dy}{dt}\right)_{c_1}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11a)$$

Using these conditions and Eq. (11), we get

$$c_1(Cc - \tfrac{1}{6}c^3) = c(\Gamma c_1 - \tfrac{1}{6}c_1^3), \quad . \quad . \quad . \quad (12)$$

$$-c_1(C - \tfrac{1}{2}c^2) = c(\Gamma - \tfrac{1}{2}c_1^2). \quad . \quad . \quad . \quad (13)$$

Solving these as simultaneous equations for C and Γ, we obtain

$$\left.\begin{aligned} C &= \tfrac{1}{6}c(c + 2c_1), \\ \Gamma &= \tfrac{1}{6}c_1(c_1 + 2c). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (14)$$

Substituting these values in Eqs. (8) and (10), the equations of the two parts of the elastic curve are obtained in their final form.

To find the maximum deflection, it is necessary to equate dy/dx and dy/dt to zero, because the deflection may reach its maximum on either side of the applied load. Equating expression (7) to zero we get for the abscissa x_m at which the deflection is a maximum:

$$x_m = \sqrt{2C} = \sqrt{\frac{c(c + 2c_1)}{3}}. \quad . \quad . \quad (15)$$

Similarly, equating expression (9) to zero gives

$$t_m = \sqrt{2\Gamma} = \sqrt{\frac{c_1(c_1 + 2c)}{3}}. \quad . \quad . \quad (16)$$

Let $c < c_1$, that is, let the load be applied at the right of the center of the span. It can be shown that in this case $x_m > c$, and $t_m < c_1$, consequently, Eq. (15) has no physical meaning, and the actual maximum takes place at the abscissa t_m determined by Eq. (16). On the other hand, if $c > c_1$, the deflection is a maximum at the distance x_m from the support B, this distance being given by Eq. (15).

Case C. A cantilever beam (Fig. 5) is uniformly loaded with a distributed load q per unit length. Here the second derivative which enters in Eq. (1) is positive, hence if the expression for the moment be also considered as positive no minus sign is needed in the equation. At a distance x from the wall, or x_1 from the free end, the bending moment is

$$M = qx_1 \cdot \frac{x_1}{2} = \tfrac{1}{2} q (l - x)^2, \quad . \quad . \quad . \quad (17)$$

so that Eq. (1) becomes

$$\frac{d^2y}{dx^2} = \frac{q}{2EI}(l - x)^2. \quad . \quad . \quad . \quad . \quad (18)$$

Integrating this expression twice, we get

$$\frac{dy}{dx} = \frac{q}{2\,EI}\left[C - \frac{(l-x)^3}{3}\right]; \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$y = \frac{q}{2\,EI}\left[Cx + \frac{(l-x)^4}{12} + C_1\right]. \quad . \quad . \quad (20)$$

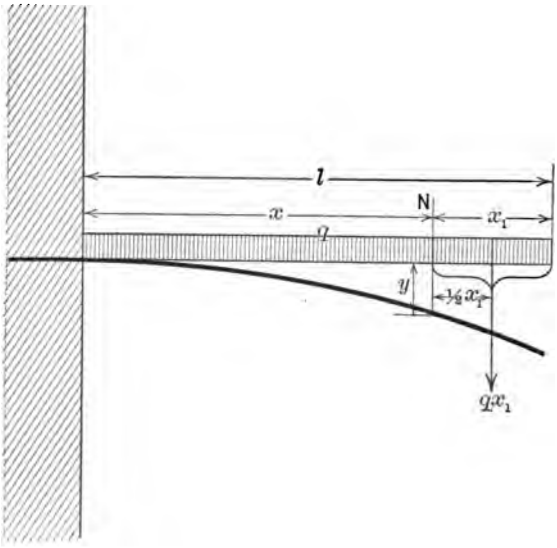

FIG. 5. — A cantilever beam uniformly loaded.

The constants of integration, C and C_1, are determined from the condition that when $x = 0$, $y = 0$, and $dy/dx = 0$. Hence

$$C - \tfrac{1}{3}\,l^3 = 0, \quad \text{and} \quad \tfrac{1}{12}\,l^4 + C_1 = 0;$$

or

$$C = \tfrac{1}{3}\,l^3; \quad C_1 = -\tfrac{1}{12}\,l^4.$$

The maximum deflection is evidently at the free end of the beam, and Eq. (20) gives

$$y_{\max} = \frac{q}{2\,EI}(Cl + C_1) = \frac{ql^4}{8\,EI}. \quad . \quad . \quad (21)$$

Case D. Let us now consider a case similar to *Case B*, except that the left-hand end of the beam (Fig. 6) is rigidly fixed in a wall, and the right-hand end rests on a support which is lower than the left-hand support by a given small amount δ. It is shown in the lower part of Fig. 6 that the wall may be replaced by a knife-edge support, provided that the beam be extended by a length m and loaded with a weight R. The values of R and m must be so chosen that the elastic line has a horizontal tangent at A. If this condition is fulfilled the two beams in Fig. 6 are subjected to the same stresses and their elastic curves are identical.

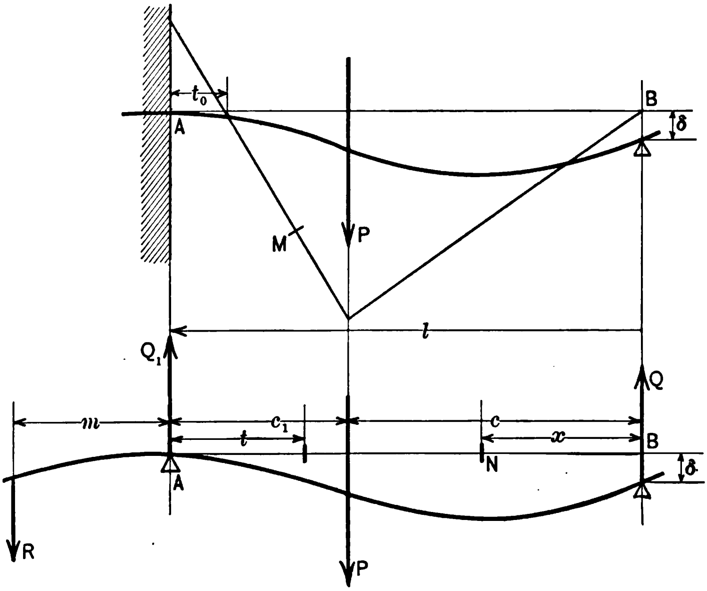

FIG. 6. — A beam fixed at one end and freely supported at the other end.

Let the unknown reactions of the supports be Q and

Q_1. Then the conditions for equilibrium of the forces applied to the beam, and for equilibrium of their moments, are:

$$\left.\begin{aligned} &P + R = Q + Q_1; \\ &R\,(m + l) + Pc = Q_1 l; \\ &Pc_1 = Rm + Ql. \end{aligned}\right\} \quad . \; . \; . \; . \quad (22)$$

The second of these equations expresses the condition that the moment of forces R and P with respect to point B, which moment tends to rotate the beam clockwise, is equal to the moment of force Q_1 which tends to rotate the beam counter-clockwise. The third equation expresses a similar condition with regard to point A. Eq. (6) for the right-hand side of the beam becomes

$$-EI\frac{d^2y}{dx^2} = Qx, \quad . \; . \; . \; . \quad (23)$$

and being integrated twice in succession gives

$$EI\frac{dy}{dx} = C - \tfrac{1}{2}Qx^2, \quad . \; . \; . \; . \quad (24)$$

and

$$EIy = Cx - \tfrac{1}{6}Qx^3 + \delta EI. \quad . \quad (25)$$

The second constant of integration is equal to δEI, because y must be equal to δ when $x = 0$.

For the left-hand side of the beam we have

$$-EI\frac{d^2y}{dt^2} = Q_1 t - R\,(m + t), \quad . \; . \; . \quad (26)$$

which, being integrated, gives

$$EI\frac{dy}{dt} = R\,(mt + \tfrac{1}{2}t^2) - \tfrac{1}{2}Q_1 t^2, \quad . \; . \quad (27)$$

and

$$EIy = R\,(\tfrac{1}{2}mt^2 + \tfrac{1}{6}t^3) - \tfrac{1}{6}Q_1 t^3 \quad . \; . \quad (28)$$

Both constants of integration are equal to zero, because for $t = 0$, $y = 0$ and $dy/dt = 0$.

Applying now Eqs. (11a) we get

$$\left.\begin{aligned} Cc - \tfrac{1}{6}Qc^3 + \delta EI &= R\left(\tfrac{1}{2}mc_1^2 + \tfrac{1}{6}c_1^3\right) - \tfrac{1}{6}Q_1c_1^3, \\ C - \tfrac{1}{2}Qc^2 &= \tfrac{1}{2}Q_1c_1^2 - R\left(mc_1 + \tfrac{1}{2}c_1^2\right). \end{aligned}\right\} \quad (29)$$

Eqs. (22) and (29) contain in five equations five unknown quantities, viz., Q, Q_1, R, m, and C, which can thus be determined. Then Eqs. (25) and (28) will give the shape of the elastic curve, and finally the abscissa corresponding to the maximum deflection can be determined from Eq. (24) by equating its right-hand member to zero. This gives

$$x_m = \sqrt{2\,C/Q}.$$

It will be noted that in the case just considered the strength of the beam (see Chapter I) can be determined only in connection with its elastic curve, because Eqs. (22) are not sufficient to determine the reactions of the supports. Eqs. (29) contain additional data for the restraining moment Rm, for the reactions Q and Q_1, and for the constant C which determines the shape of the elastic curve.

The bending moment M varies along the beam as is shown by the broken line in the upper part of Fig. 6. It has a positive maximum at the point of application of the load, and a negative maximum at the left-hand support. We have

$$\left.\begin{aligned} M_{x=c} &= Qc, \\ M_{t=0} &= -Rm. \end{aligned}\right\} \quad \ldots\ldots \quad (30)$$

Either of these moments may be larger in absolute value than the other, according to the position of the load P; the dangerous cross section may be either at P, or at A. The elastic curve has a point of inflection at a point for which the bending moment is zero. According to Eq. (26), the abscissa t_0 of this point is determined from the condition

$$Q_1 t_0 = R(m + t_0). \quad . \quad . \quad . \quad . \quad (31)$$

To the right of this point the beam is concave on the upper side, and to the left it is concave on the lower side.

Prob. 1. — Deduce expressions corresponding to Eq. (5), for the cases in which the load is uniformly distributed, (*a*) over the whole span, and (*b*) over a portion of the span.

Prob. 2. — For a beam freely supported on both ends draw to scale an elastic curve produced by a uniformly distributed load. On the same curve sheet draw an elastic curve for the same beam when a concentrated load is applied at the center of the span. Select the loads in such a way as to have the same maximum deflection in both cases. Use a much larger scale for ordinates than for abscissæ. Explain from a physical point of view, the reason for the difference in the shape of the two curves.

Prob. 3. — Deduce an expression for the elastic curve when two concentrated loads are applied to the beam, as in Fig. 1.

Prob. 4. — Modify Case B by adding a load uniformly distributed over the span.

Prob. 5. — Modify Case B by adding a load uniformly distributed over a portion of the span. Begin with the simplest case in which the distributed load extends from one support to the point of application of the concentrated load.

Prob. 6. — Deduce an expression for $y_{\max}$ for a cantilever beam loaded with a concentrated load at the free end.

Prob. 7. — Extend Case C by adding a concentrated load at a distance c from the wall.

Prob. 8. — Modify Case C by assuming the beam to be fixed in the wall not horizontally, but at a small angle α upward.

Prob. 9. — A concentrated load travels along a cantilever beam. Deduce an expression for the deflection at the end of a beam as a function of the position of the load.

Prob. 10. — Make all the possible simplifying assumptions in Case D, and write, in their final form, formulæ for the elastic curve and for the reactions of the supports. *Hint.* — Take $\delta = 0$, and take the load at the center of the span.

Prob. 11.— In Case D, assuming δ to be equal to zero, find a position of the load such that the reactions of the two supports will be equal.

Prob. 12. — A horizontal beam is fixed at both ends and is loaded with a concentrated weight at the center of the span. Find the equation of the elastic curve.

Prob. 13. — Solve Prob. 12 when the load is not at the center of the span.

Prob. 14. — Solve Prob. 12 when the load is uniformly distributed, (*a*) over the whole span, (*b*) between a support and a given point in the span.

Prob. 15. — In Probs. 11 to 14 find the position of the point of inflection.

Prob. 16. — In Fig. 6 let δ be zero when the beam is not loaded, and let the support B be elastic, so that when the beam is loaded δ is proportional to Q. Deduce the equation of the elastic curve, and write the equations from which the constants are to be determined.

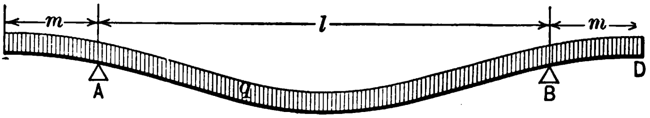

FIG. 7. — A beam with projecting ends, uniformly loaded.

Prob. 17. — A beam is resting on two supports (Fig. 7), and extends over each support by an amount m. It is uniformly loaded over its whole length; find the equation of the elastic curve.

Prob. 18. — For the beam in Prob. 17 find the points of inflection and the points where the moment is a maximum.

Prob. 19. — Solve Prob. 17 when the load, instead of being distributed, is concentrated at C and D, and at some intermediate point between A and B. Take all the three loads different.

CHAPTER III.

FATIGUE OF MATERIALS.

Let a steel bar or rod be placed in a testing machine and subjected to a slowly and gradually increasing tension until it breaks. Let 30 tons be required to rupture the bar; this figure is called its ultimate static strength. The same bar may be broken by a tension of only about 15 tons, provided that the force is applied a great many times and that it varies between zero and 15 tons. With such a fluctuating load, peculiar molecular changes take place in the metal which reduce its strength. This phenomenon is known as fatigue in the metal. The bar may be broken by a load of only 10 tons, if this load varies a great many times in succession between 10 tons tension and 10 tons compression, in other words, between +10 tons and −10 tons.

The effect of fatigue on the ultimate strength of steel is represented graphically in Fig. 8. The ordinates of the curve *ABC* represent the breaking force *P* as a function of the range of fluctuations of the load; the abscissæ give the lower limit *p* of the fluctuations of the applied force. Thus, at *A*, which corresponds to, say, 30 tons, the load is stationary, $p = P$, or $Oa = aA$, and point *A* is found as intersection of the curve with the straight line *OA* inclined at an angle of 45 degrees to the axis of abscissæ. At an intermediate point, such as *D*, the bar breaks under a load *dD*, when it fluctuates between *Od* and *dD*. At point *B* the load fluctuates between zero and *OB*; this corresponds to 15 tons in our example. At *G* the bar is subjected to alternate tension *gG* and compression *Og*, the total range of variation of stress being *Gg* + *Og*. By drawing line *OC* at an angle of 45 degrees to the axis of abscissæ the point *C* of the curve is found, for which the tension *Cc* and the compression *Oc* are equal; in our example *cC* = *Oc* = 10 tons. Parts of the curve outside the range *CA* do not enter into the discussion in this chapter.

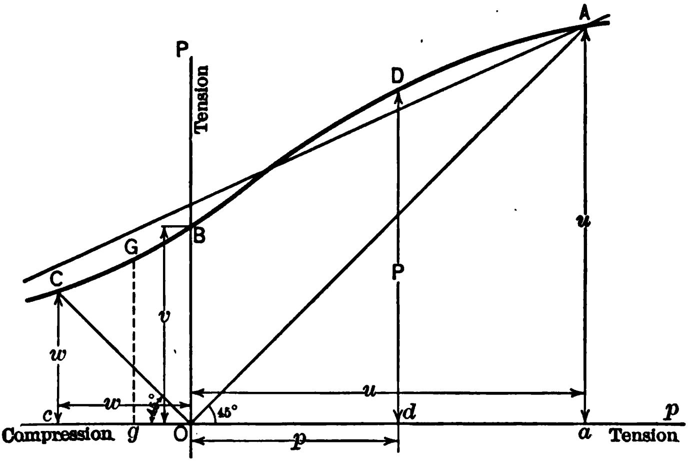

FIG. 8. — Curve of ultimate strength and the effect of fatigue.

To know the shape of the curve *ABC* for a given material is of the utmost importance in the design of machine parts or of a bridge subject to variable stresses and shocks. Experimental data are comparatively few and difficult to get; moreover they cover only a few points on the curve; hence the numerous attempts of various investigators to establish an empirical equation of the curve *ABC*. The purpose of this chapter is to show the reasoning which leads to these formulæ; this topic furnishes an excellent example of approximate empirical relations established for use in engineering practice, where exact data or theory are lacking.

Launhardt's Formula.

Launhardt's formula covers only part AB of the curve, where the maximum stress P and the minimum stress p are both tension or both compression. Let the ordinate aA be denoted by u, and the ordinate OB by v. Launhardt makes a reasonable assumption that the increase of P over v is directly proportional to the ratio $P : p$, or

$$P - v = k\frac{p}{P}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where k is a constant. For the point A, $P = p = u$, so that the above equation gives for the value of k

$$u - v = k.$$

Substituting this value of k in Eq. (1), we get

$$P = v + (u - v)\frac{p}{P}. \quad . \quad . \quad . \quad . \quad (2)$$

This is Launhardt's formula for stresses of the same kind. For practical purposes it is convenient to leave the formula in this form, and not solve it for either p or P, because the ratio $p : P$ is known before the stresses themselves are determined. As the value of P represents the breaking strength, to obtain the permissible *working* stress, P must be divided by a factor of safety, which varies from 3 to 10, according to the importance of the part, the material used, and the certainty with which the stresses can be determined.

Weyrauch's Formula.

By a reasoning similar to that given above, Weyrauch deduced a formula applicable to the part BC of the curve, in other words, when the stress varies between a tension P and a compression p. Let the ordinate cC be denoted by w. Weyrauch assumes that the decrease in the value of P below that of v is proportional to the value of the ratio $p : P$, or, analogously to Eq. (2),

$$P = v - (v - w)\frac{p}{P}, \quad . \quad . \quad . \quad . \quad (3)$$

wherein p is understood to represent the absolute value of the stress, so that the ratio $p : P$ is always positive and less than unity.* For the point B, $p = 0$ and $P = v$; for the point C, $p = P$ and $P = w$. For other points, P has values intermediate between v and w.

Johnson's Formula.

An objection to the foregoing two formulæ is that each of them covers only a part of the useful range of the curve ABC, so that there is a break in the slope of the curve at B; in fact, the curve ABC is supposed to consist of two separate curves, merely having a common ordinate v at B. It is more reasonable to assume that both branches are parts of the same curve; consequently, several formulæ have been proposed to cover the whole range ABC. The late Prof. J. B. Johnson assumed that, in view of inadequate knowledge concerning the values of v and w, it is sufficient at present to replace the curve ABC by a straight line. This line must pass through A, since the value of u is known with considerable certainty. The slope of the line must be such as to satisfy as far as possible the known values of v and w.

---

* If it is preferred to consider P negative when it represents compression, Weyrauch's formula must be written in the form $P = v + (v - w)\,p/P$.

The general equation of a straight line between P and p is

$$P = ap + b. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

As in the two preceding formulæ, it is desirable to express P as a function of the ratio $p : P$. Dividing both sides of Eq. (4) by P, we have

$$1 = a\left(\frac{p}{P}\right) + \frac{b}{P},$$

or, solving for P,

$$P = \frac{b}{1 - a\left(\frac{p}{P}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The coefficient b can be eliminated because of the condition that $P = u$ when $p : P = 1$. Thus, applying Eq. (5) to point A, we get

$$u = \frac{b}{1 - a};$$

hence

$$P = \frac{(1 - a)\,u}{1 - a\left(\frac{p}{P}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This is Johnson's formula in its most general form. The value of a must be determined experimentally. For steel, so far as is known, v is equal to about $\frac{1}{2}u$; the substitution of this value in Eq. (6), when $p = 0$, gives

$$\tfrac{1}{2}u = (1 - a)\,u,$$

or $a = \frac{1}{2}$. Hence, for steel,

$$P = \frac{\frac{1}{2}u}{1 - \frac{1}{2}(p/P)} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

This is the form in which Johnson's formula is usually given. This formula gives a value for w of $\frac{1}{3}u$, because $P = \frac{1}{3}u$ when $p : P = -1$.

Unwin's Formula.

Unwin assumes that the curve ABC forms part of a parabola of the second degree between p and P. The general equation of such a parabola may be written in the form

$$(mP + np)^2 + u(rP + sp) + u^2 f = 0, \quad . \quad . \qquad (8)$$

where m, n, r, s, and f are coefficients to be determined from experimental results. The second and the third terms are multiplied by known quantities u and u^2, respectively, in order to preserve the homogeneity of the equation and to make the coefficients m, n, r, s, and f abstract numbers (numerics).

Instead of assuming that P depends on the ratio $p : P$, as in the preceding formulæ, Unwin assumes that P depends on the range $P - p$ of the fluctuation in stress. Let $P - p$ be denoted by Δ. Then

$$mP + np = mP + nP - nP + np = (m + n)P - n\Delta;$$
$$rP + sp = rP + sP - sP + sp = (r + s)P - s\Delta.$$

Eq. (8) takes the form

$$(m+n)^2P^2 + n^2\Delta^2 - 2n(m+n)P\Delta + u(r+s)P - us\Delta + u^2f = 0.$$

This is an ordinary quadratic equation with respect to P. Being solved it gives

$$P = Q\Delta + Ru + S\sqrt{u^2 - Vu\Delta}, \quad . \quad . \qquad (9)$$

where Q, R, S, and V are certain combinations of the coefficients m, n, r, s, and f.

If any four points on the curve *ABC* are known from tests, the coefficients Q, R, S, and V may be determined by substituting the known values of P and Δ in Eq. (9) and solving the four equations for these coefficients. According to the information available at present the results of many experiments check satisfactorily with formula

$$P = \tfrac{1}{2}\Delta + \sqrt{u^2 - Vu\Delta}, \quad . \quad . \quad . \quad (10)$$

in other words, in formula (8), $Q = \tfrac{1}{2}$, $R = 0$, $S = 1$. The value of V is about 1.5 for ductile iron and steel. For hard materials, Unwin advises the use of 2 for V. With more knowledge of the properties of materials, the values of Q, R, S, and V in Eq. (9) will be determined with greater precision.

Other Possible Formulæ.

For several reasons, it does not seem advisable to use the complicated formula (9) to express the curve *ABC*. The formula is unsymmetrical, and the ultimate strength P is expressed through the range of fluctuations $P - p = \Delta$, while it is much more convenient for practical purposes to express it through the ratio of fluctuations $p : P$. A wide variety of possible shapes of the curve *ABC* may be covered by slightly modifying the simple formulæ (1), (3), or (5). Thus, if experience should show that Eq. (1) does not represent the behavior of a metal with sufficient accuracy, a quadratic term may be added on the right-hand side, so that the formula becomes

$$P - v = k_1\left(\frac{p}{P}\right) + k_2\left(\frac{p}{P}\right)^2. \quad . \quad . \quad . \quad (11)$$

With a proper selection of the coefficients k_1 and k_2, almost any degree of curvature may be obtained. Weyrauch's formula may be modified in the same way.

It is also possible to combine Launhardt's and Weyrauch's formulæ by using a series of the form

$$P = A + B\left(\frac{p}{P}\right) + C\left(\frac{p}{P}\right)^2 + \text{etc.} \quad . \quad . \quad (12)$$

A few such terms will represent the results of experiments with sufficient accuracy.

Johnson's formula (5) may be made more flexible and adaptable in several ways, for instance by adding an extra term in the numerator, or in the denominator, or in both. These, being merely correction terms, need not be calculated with extreme precision. We thus obtain the following empirical formulæ:

$$P = \frac{b \pm c\left(\frac{p}{P}\right)}{1 - a\left(\frac{p}{P}\right)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$P = \frac{b}{1 - a\left(\frac{p}{P}\right) \pm f\left(\frac{p}{P}\right)^2}. \quad . \quad . \quad . \quad (14)$$

Various other modifications may be introduced by using higher powers of $p : P$. The coefficients which occur in these formulæ must be determined by using a few known experimental values of P. Thus, for instance, let it be required to determine the coefficients a, b and c in formula (13), when the only experimental data available are the values of u, v and w. It is convenient in this case to express v and w as known fractions of u, thus

$$v = mu; \quad w = nu. \quad . \quad . \quad . \quad . \quad (15)$$

Substituting these values in succession in Eq. (13) we obtain three equations

$$\left.\begin{aligned} u &= \frac{b+c}{1-a}; \\ mu &= b; \\ nu &= \frac{b-c}{1+a}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

These equations can be easily solved for a, b and c, and the results used in Eq. (13) for obtaining intermediate values of P. Eqs. (11), (12) and (14) may be treated in a similar manner. If the number of known points on the curve is larger than the number of coefficients in the empirical formula, the most probable values of the coefficients may be determined by the method of least squares or by trial. See Part III (Thermodynamics), Chapter III.

Prob. 1. — Plot values of P according to Launhardt's and Weyrauch's formulæ for $v = \frac{1}{2}u$ and $w = \frac{1}{3}u$. Also draw the straight line corresponding to Johnson's formula, and determine graphically maximum per cent discrepancy between the curves and the straight line.

Prob. 2. — Check the above maximum per cent discrepancy analytically, by equating to zero the first derivative of the analytical expression for per cent discrepancy.

Prob. 3. — There is a discontinuity in the slope where Launhardt's curve passes over to Weyrauch's curve. Determine analytically the angle between the two curves.

Prob. 4. — Show that if $u + w = 2v$, Launhardt's and Weyrauch's formulæ represent parts of the same curve.

Prob. 5. — A certain kind of steel breaks at a steady tension of 40 kg. per sq. mm. In order to break a similar sample by a tension of 20 kg. per sq. mm., tension must alternate with a compression of 10 kg. per sq. mm. Write Johnson's formula for this material.

Prob. 6. — Plot curves ABC according to Unwin's formula (10) for ductile steel and for hard steel. Draw straight lines that agree as nearly as possible with these curves and calculate the corresponding values of a in formula (6). Assume u to have the same value in all four cases, the four curves passing through the same point A.

Prob. 7. — For a certain material $v = 0.62\,u$ and $w = 0.45\,u$. Can the properties of the material be satisfactorily represented by formula (10), and if not, what will be the numerical coefficients in the more general formula (9)?

Prob. 8. — Using relation (15) determine the values of the coefficients in formula (12), in terms of m, n and u.

Prob. 9. — Using relation (15) determine the values of the coefficients in formula (13), in terms of m, n and u.

Prob. 10. — Using relation (15) determine the values of the coefficients in formula (14), in terms of m, n and u.

Prob. 11. — From the available literature on the subject, plot experimental fatigue curves, such as ABC, for various materials, and investigate which of the formulæ quoted in this chapter can be made to express the results of the tests most accurately.

CHAPTER IV.

EULER'S THEORY OF FLEXURE OF LONG RODS.

A LONG elastic rod or flat spring (Fig. 9) is pivoted at both ends and is loaded with a force P in such a way that it is bent out of its original straight shape. A guide g constrains the upper end of the rod to remain on the original vertical line. The problem is to find a relation between the force P and the deflection a which it produces. According to the general theory of flexure, which the student can find in any standard work on mechanics of materials, we have, for any cross section of the rod such as A,

$$\frac{EI}{\rho} = M. \qquad \ldots\ldots\ldots \qquad (1)$$

In this expression M is the bending moment of the external forces with respect to the cross section under consideration, in kg.-cm.; E is the modulus of elasticity of the material, in kg. per sq. cm.; I is the moment of inertia of the cross section with respect to the axis perpendicular to the plane of Fig. 9 and passing through the center of gravity of the cross section, in cm.^{4}; ρ is the radius of curvature of the axis of the rod at A, in cm.

When the curvature is small ρ is large and we have approximately (see Prob. 10 below)

$$\frac{1}{\rho} = -\frac{d^2y}{dx^2},$$

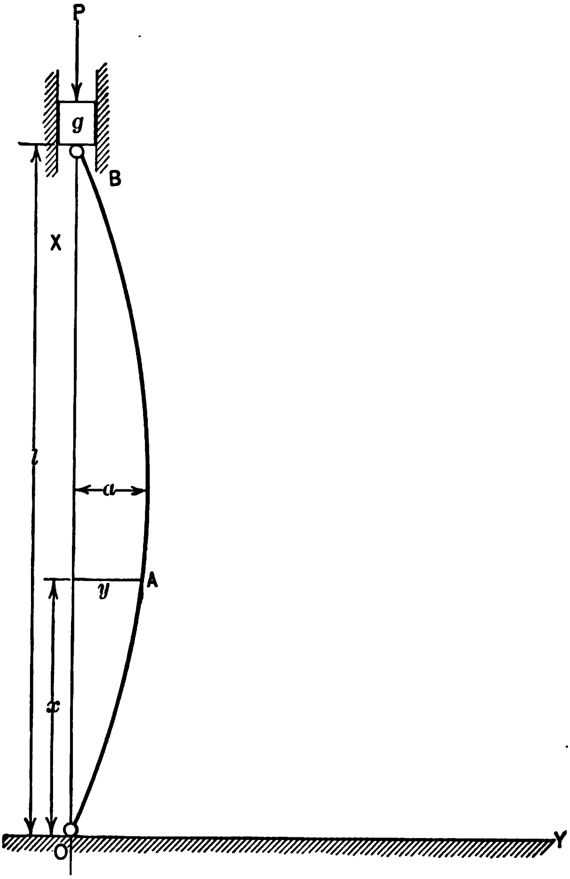

FIG. 9. — A long rod subjected to endwise compression.

so that Eq. (1) becomes

$$-IE\frac{d^2y}{dx^2} = M.$$

In the case under consideration the bending moment $M = Py$, so that

$$-IE\frac{d^2y}{dx^2} = Py. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or, denoting, for the sake of abbreviation,

$$\sqrt{\frac{P}{IE}} = n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

we have

$$\frac{d^2y}{dx^2} = -n^2y. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This is the differential equation of flexure, and it shows that y is such a function of x that its second derivative with respect to x is proportional to the function itself, the coefficient of proportionality $-n^2$ being negative. A sine (or cosine) function of the general form

$$y = a \sin (bx + c) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

is the only function which satisfies this condition.* In this formula, a, b, and c are certain constants to be determined from the given conditions of the problem. Eq. (5) means that the axis of the rod is bent into a sine curve. To find its exact shape we differentiate Eq. (5) twice:

$$\frac{dy}{dx} = ab \cos (bx + c). \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{d^2y}{dx^2} = -ab^2 \sin (bx + c). \quad . \quad . \quad . \quad (7)$$

Substituting the values from Eqs. (5) and (7) into Eq. (4) and cancelling the sine term, we obtain

$$-ab^2 = -n^2a,$$

or

$$b = n = \sqrt{\frac{P}{IE}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

---

* If the right-hand side of Eq. (4) had a plus instead of a minus sign, the simplest solution would be of the form $y = fe^{-kx}$. For a complete integration of Eq. (4) see the Note at the end of the chapter.

The constant c is determined from the terminal conditions of the problem; namely, when $x = 0$ or $x = l$, y must be equal to zero. Hence

$$\sin c = 0 \quad \text{and} \quad \sin (bl + c) = 0,$$

or, in the simplest case, $c = 0$ and $bl = \pi$. This shows that the rod forms a complete half-wave of a sinusoidal curve. Using the value of b from Eq. (8), we get

$$l\sqrt{\frac{P}{IE}} = \pi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8a)$$

or

$$P = \frac{\pi^2 EI}{l^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The amplitude a of the sine-wave in Eq. (5) remains indefinite, because Eq. (2) is satisfied with the solution for any value of a whatsoever, and there are no other physical or mathematical conditions from which a could be determined. This result may be taken to indicate that, as soon as the force P reaches the critical value, Eq. (9), the rod may bend to any value of deflection a, once it has started to bend due to the slightest lack of symmetry. One must remember, however, that a must be small compared to l, because of the approximate value for ρ used in Eq. (2), which is true for small curvatures only.*

For practical purposes Eq. (9) determines the breaking load P, because no part of a structure can be considered safe, in which such an indefinite buckling may take place. It is convenient to know the crushing force per unit area of cross section of the rod. Designating this area by A, and dividing both sides of Eq. (9) by A, we have

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* For a differential equation of flexure of columns more accurate than Eq. (4) see R. F. Deimel, The Theory of Columns, Bulletin of the Society for the Promotion of Engineering Education, Vol. 6, p. 258, December 1915. It is shown there that for each load there is a definite deflection. This is also confirmed by experimental investigations.

$$\frac{P}{A} = \frac{\pi^2 E}{l^2} \cdot \left(\frac{I}{A}\right).$$

But the moment of inertia I, being of the dimension (length)^4, may be represented as

$$I = Ar^2,$$

where the fictitious length r is called the radius of gyration of the cross section; thus, we finally get the well-known theoretical formula for the flexure of long rods

$$\frac{P}{A} = \frac{\pi^2 E}{(l/r)^2}. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

In the preceding theory the rod is supposed to be hinged at both ends. Three other cases of flexure are shown in Figs. 10, 11, and 12. The first figure shows a rod fixed at the lower end B, and loaded at the free end A. This case could be considered in a way similar to that outlined above, only the terminal conditions for the determination of the constants of integration are different. However, the result may be foreseen directly by imagining the rod extended downward, as shown by the dotted line. The case is reduced to that in Fig. 9 and formula (9) is applicable, if l is understood to be the total length of the imaginary rod between A and C; or $l = 2\,L$, where L is the actual length of the rod.

In a similar manner, the rod in Fig. 11, fixed at both ends, B and H, may be extended both ways, by one-quarter of the sine-wave, so that we have three rods, CD, DE, and EF, one on top of another, bent according to Fig. 9. Eq. (9) is again applicable if the theo-

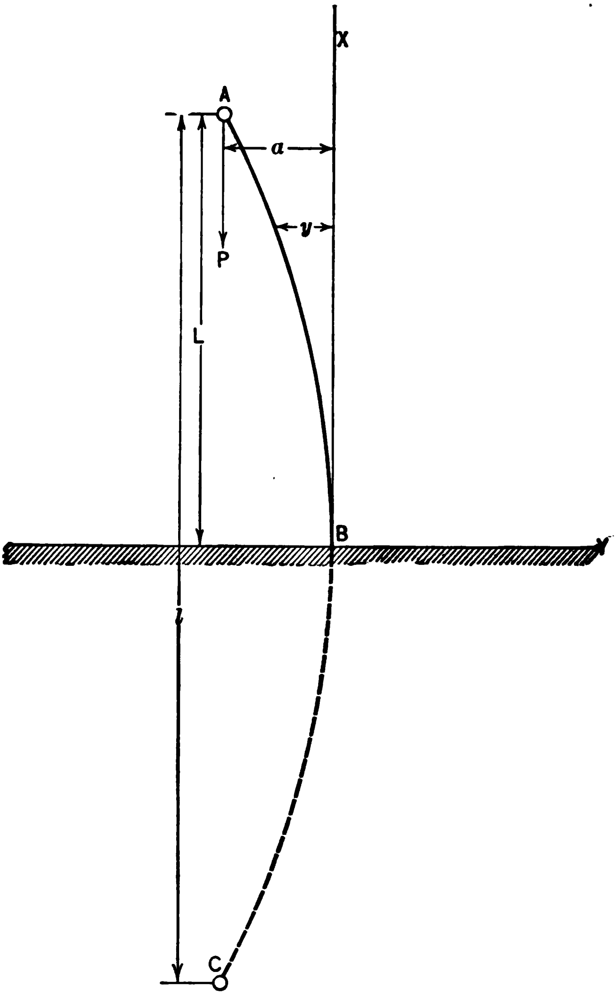

FIG. 10. — A long vertical rod fastened at the lower end and loaded at the upper end.

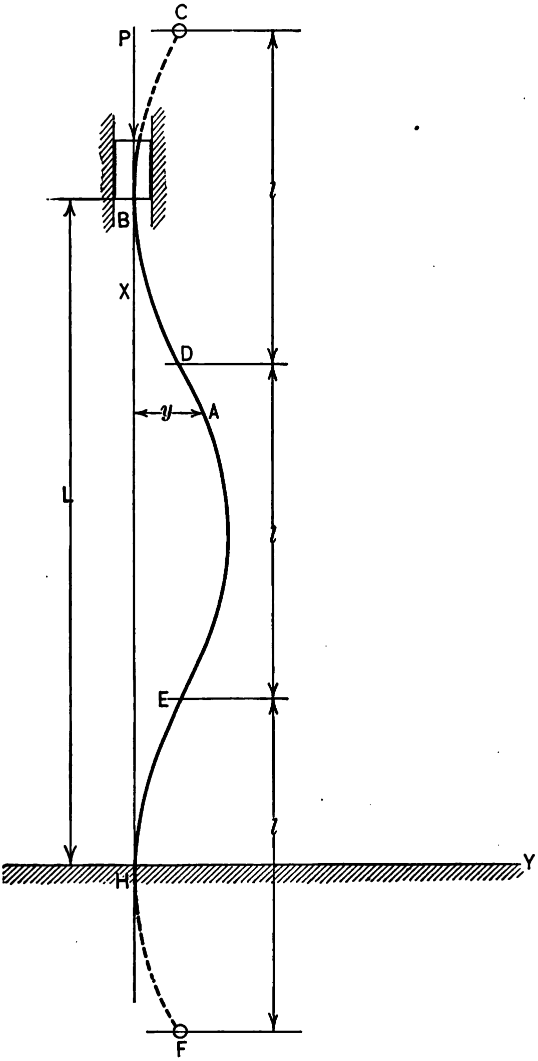

FIG. 11. — A long rod fastened at both ends and subjected to a vertical load.

retical length l be taken equal to $\frac{1}{2} L$, where L is the actual length of the rod.

The foregoing theory is applicable only to very long thin rods, for which the direct compressive action of the load P is negligible as compared with the bending moment produced by a deflection. For ordinary shorter columns and for struts used in bridge construction, various semi-empirical formulæ are used. The elastic curve of a short column is not a sine wave, so that the relations between L and l deduced above do not exactly hold. The strength of such columns is discussed in the next chapter.

Note. — To find the general solution of Eq. (4), multiply both sides of it by dy/dx. This gives

$$\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right] = -n^2\frac{d}{dx}(y^2), \quad \ldots\ldots \quad (11)$$

or

$$\left(\frac{dy}{dx}\right)^2 = -n^2y^2 + n^2a^2. \quad \ldots\ldots \quad (12)$$

The term n^2a^2 represents the constant of integration, a being so far an arbitrary quantity. The constant could be simply written C, but in physical problems it is of advantage to give all constants a certain physical meaning. Thus, y being a length, a has also the dimension of a length. When $dy/dx = 0$, $y = a$, so that a is the maximum deflection of the rod. We have further

$$\frac{dy}{dx} = \pm n\sqrt{a^2 - y^2}. \quad \ldots\ldots\ldots \quad (13)$$

The double sign means in this case that the rod may buckle either to the positive or to the negative side of the vertical axis. Limiting our attention to the case shown in Fig. 9 we need retain only the plus sign. Hence

$$dx = \frac{dy}{na\sqrt{1 - \left(\frac{y}{a}\right)^2}},$$

or, after integration,

$$x = \frac{1}{n}\left[\sin^{-1}\frac{y}{a} - c\right], \quad \ldots\ldots\ldots \quad (14)$$

where c is a new constant of integration. Solving this equation for y, expression (5) is obtained; it will be remembered that b is equal to n, according to Eq. (8).

Prob. 1. — Assuming the crushing load in the case indicated on Fig. 9 to be 100 per cent, what are the values of crushing load for identical rods of the same actual length L, but fastened as shown in Figs. 10 and 11?

Prob. 2. — A spring of length L is bent as shown in Fig. 12, forming N sine waves (six half-waves in the particular case shown in the figure). How many times larger force P is required to hold the spring in this position, as compared with the force necessary to hold the same spring in the shape shown in Fig. 9?

Prob. 3. — For round rods formula (9) may be represented in the form

$$P = \text{Const.} \left(\frac{d^2}{l}\right)^2,$$

where d is the diameter of the cross section. Find the numerical value of the constant for steel rods, d and l being in cm. and P in kg. The modulus of elasticity E for steel is about 2 million kg. per sq. cm.

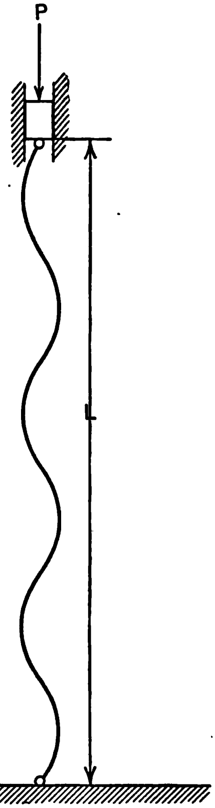

Fig. 12. — A long spring held by an end force so as to form six half-waves.

Prob. 4. — Compare the crushing load on a round rod to that on a rod having a square cross section of the same area. If the prescribed crushing load be the same in both cases, which rod is lighter and by how much?

Prob. 5. — A flat rod of cross section $a \times b$ (Fig. 13) is to be strengthened by two ribs of thickness c, so as to be as rigid against buckling in the direction XX as it is in the direction YY. Determine the height z of the ribs.

Prob. 6. — Deduce the expressions for the radius of gyration of some simple cross sections used in practice, such as are shown in Fig. 14.

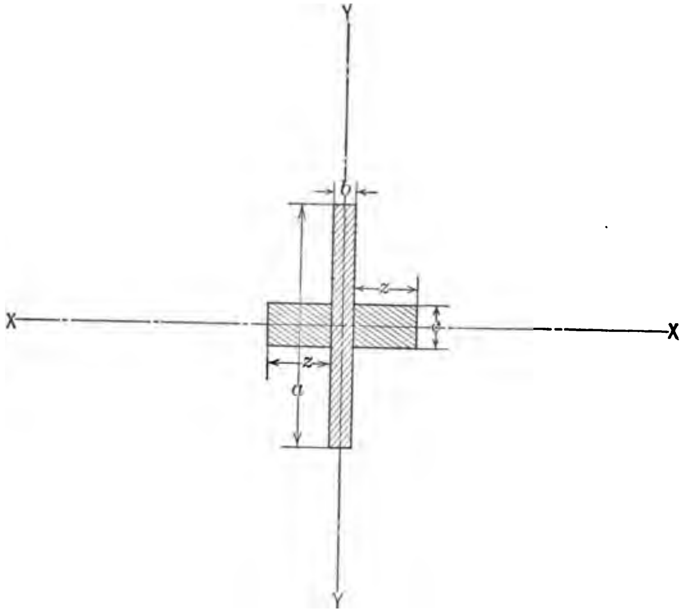

FIG. 13. — A rectangular cross section strengthened by ribs.

Prob. 7. — Sketch the general shape of the curve which gives the unit stress P/A as a function of l/r. Why does the curve seem to indicate that for small values of l the rod can apparently support an infinite load, which is of course not the case?

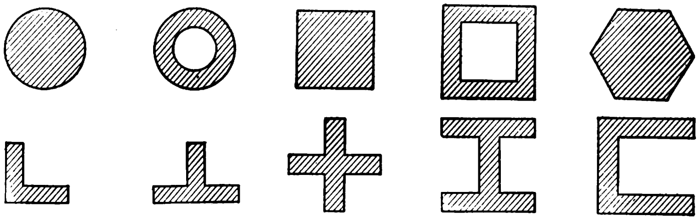

FIG. 14. — Cross sections of columns used in practice.

Prob. 8. — Deduce an expression similar to Eq. (9) for the case shown in Fig. 10; use a differential equation similar to

Eq. (2), and apply the proper terminal conditions for $x = 0$ and $x = L$. The result must check with that anticipated in the text above by reducing this case to that shown in Fig. 9. *Hint.* — The bending moment is $P(a + y)$, and one of the terminal conditions is that $dy/dx = 0$ when $x = 0$.

Prob. 9. — Deduce an expression similar to Eq. (9) for the case shown in Fig. 11; use a differential equation similar to Eq. (2), and apply the proper terminal conditions at the ends. The result must check with that obtained directly from Eq. (9) by putting the effective length l equal to one-half the actual length L of the rod. *Hint.* — For a point on the rod, such as A, the bending moment is equal to $Py - M_0$, where M_0 is the so-called moment of fixture. This is the moment exerted at the upper end of the rod to keep it vertical. The moment of the force P tends to turn the upper part of the rod counter-clockwise, while the moment M_0 tends to turn it clockwise; for this reason M_0 enters the equation with the minus sign, if Py enters with the plus sign. The terminal conditions are: at $x = 0$, $y = 0$ and $dy/dx = 0$; at $x = L$, $dy/dx = 0$. There are, therefore, three equations from which M_0 and the two constants of integration may be evaluated.

Prob. 10. — Deduce an expression similar to Eq. (9) for the case when the upper end of the rod is hinged as in Fig. 9, and the lower end is fastened as in Fig. 11; use a differential equation similar to Eq. (2), and apply the proper terminal conditions at the ends. Consult the two preceding problems.

Prob. 11. — Prove that when the curvature of the rod is small (Fig. 9), it is approximately equal to d^2y/dx^2. *First proof.* — By definition, the curvature $1/\rho = d\alpha/ds$, where α is the angle which the tangent to the curve at a point makes with the X axis, and ds is an element of the curve. In general, $\tan \alpha = dy/dx$, but in the case under consideration the angle is small throughout so that approximately $\alpha = dy/dx$. Besides, ds is practically equal to dx, so that $1/\rho = d\,(dy/dx)/dx = d^2y/dx^2$. *Second proof.* — Take the general expression for the radius of curvature given in the differential calculus, and let $dy/dx = 0$.

Prob. 12. — In addition to the end load P (Fig. 9) the rod is subjected to a uniformly distributed horizontal load of p kg. per cm. length. Write down the differential equation for the bending moment and integrate it. *Solution.* — The total horizontal load is pl, and its reaction at each end of the rod is $\frac{1}{2}\,pl$. The moment of the upper reaction with respect to point A is equal to $\frac{1}{2} pl (l - x)$. The load itself over the upper portion of the rod is $p (l - x)$; and its center is distant by $\frac{1}{2} (l - x)$ from A. Thus, the moment of the load is $- \frac{1}{2} p (l - x)^2$. Therefore, the right-hand member of Eq. (2) must be supplemented by the term

$$\tfrac{1}{2} pl (l - x) - \tfrac{1}{2} p (l - x)^2 = \tfrac{1}{2} px (l - x).$$

Eq. (2) is then modified to one of the general form

$$\frac{d^2y}{dx^2} + n^2y = -hx (l - x), \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where n and h are known coefficients. According to the general theory of linear differential equations with constant coefficients we find that

$$y = (A \sin nx + B \cos nx) - \frac{h}{n^2}\left(lx - x^2 + \frac{2}{n^2}\right), \quad . \quad (16)$$

where the first expression in the parentheses on the right-hand side is the "complementary function," and the second expression is a "particular solution." A and B are the constants of integration, and are determined from the terminal conditions, as before.

Prob. 13. — Prove by a direct differentiation that solution (16) satisfies Eq. (15).

Prob. 14. — With reference to Eq. (8a), show how the value for P is obtained corresponding to Fig. 12, by using $m\pi$ in place of π, where m is an integer.

CHAPTER V.

FLEXURE OF COLUMNS.

Euler's theory of flexure given in the preceding chapter is only applicable to rods, columns, springs, etc., whose length is several hundred times greater than the transverse dimensions (radius of gyration). The reason is that in Euler's theory the direct compression is left entirely out of consideration. This compression becomes more and more important as the rod becomes stiffer and shorter, that is, as its length becomes smaller in proportion to its transverse dimensions.

For short blocks the carrying capacity, indicated by formula (9) in the preceding chapter, tends to become infinite, while in reality it is evidently limited by the ultimate compressive strength P/A of the material. Various empirical and semi-empirical formulæ are used by engineers for calculating the strength of columns of medium length, which are too short to be computed according to Euler's formula, and at the same time are too long to be considered simply as blocks subjected to a uniform compression. Some of these formulæ and their deductions are given below, as examples of reasoning by which practical problems are solved approximately, where the true physical relations are too complicated. The student will also get some experience in fitting empirical formulæ to experimental curves.

(1) *Johnson's Straight-line Formula.* — Euler's theory leads to the following relation between the crushing load per unit area P/A, and the ratio l/r:

$$\frac{P}{A}=\frac{\pi^2 E}{(l/r)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

See formula (10) in the preceding chapter. It will be seen that the crushing stress depends not upon the absolute length of the rod or column, but upon the ratio of the length to the radius of gyration of the cross section. Plotting the values of P/A against l/r as abscissæ a curve is obtained of the general shape shown in Fig. 15. This curve gives absurd values of P/A

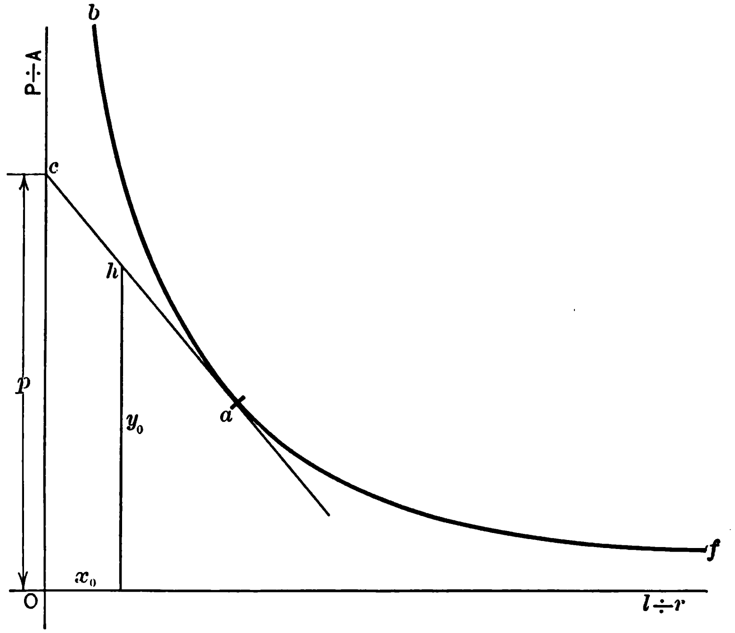

FIG. 15. — Euler's theoretical curve and Johnson's straight-line correction.

for small values of l/r, as is explained above. Thomas H. Johnson proposed therefore to replace the upper part *ab* of the curve by a straight line *ac* tangent to it. This straight line is determined by the condition that for very short rods, that is, when $l = 0$, the unit stress must be equal to the ultimate crushing strength p of the material.

The equation of this straight line is found analytically as follows: Let, for the sake of abbreviation,

$$\frac{l}{r} = x; \quad \frac{P}{A} = y.$$

We have then for Euler's curve

$$y = \frac{\pi^2 E}{x^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and for the straight line

$$y = p - kx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where k is a coefficient to be determined from the condition that the straight line is tangent to the curve (2). Let the coördinates of the point of tangency be x_1 and y_1. Then, at this point the values of y from Eqs. (2) and (3) must be equal to each other as must also the values of dy/dx. Hence

$$\frac{\pi^2 E}{x_1^2} = p - kx_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$-\frac{2\,\pi^2 E}{x_1^3} = -k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From these two equations k and x_1 can be determined. Substituting the value of k from Eq. (5) into Eq. (4) gives

$$\frac{3\,\pi^2 E}{x_1^2} = p,$$

or

$$x_1 = \left(\frac{l}{r}\right)_1 = \pi\sqrt{\frac{3E}{p}}. \quad . \quad . \quad . \quad . \quad (6)$$

Using this value of x_1 in Eq. (5) we get

$$k = \frac{2p}{3\pi}\sqrt{\frac{p}{3E}}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and Johnson's formula becomes

$$\frac{P}{A} = p - \left(\frac{l}{r}\right)\cdot\frac{2p}{3\pi}\sqrt{\frac{p}{3E}}. \quad . \quad . \quad . \quad . \quad (8)$$

This formula should be used only between the values $l/r = 0$, and l/r determined by the relation (6). For longer columns Euler's formula (1) is valid.

(2) *Hodgkinson's Formula.* — Hodgkinson was one of the first experimenters on the strength of long rods, and he found that the crushing load does not vary exactly as required by Euler's formula (1). He therefore modified it into an empirical formula

$$\frac{P}{A} = C\cdot\frac{r^m}{l^n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where the coefficients C, m, and n are to be so selected for each material, or for each kind of columns, as to satisfy the results of available experiments. Formula (9) is not now used, since the experimental data on columns can be better expressed by more rational formulæ given below. But this old formula deserves some attention, because in quite a number of engineering problems a satisfactory agreement with experimental results is obtained by the use of fractional exponents, in place of integers required by a simplified theory.

(3) *Rankine's Formula.* — Side by side with empirical formulæ, several prominent engineers, among

others Navier, Rankine, and Ritter, worked on a rational theory of flexure of short columns. The theory now accepted is as follows: A column of moderate length fails under the combined stress caused by direct compression and by flexure. The uniform stress due to the direct compression is P/A, while the maximum compressive unit stress p_1 caused by flexure on the concave side (Fig. 2) is

$$p_1 = \frac{Mz}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

In this formula M and I have the same meaning as in the preceding chapter, and z is the distance from the center of gravity of a cross section to the extreme compressed fiber.* The maximum bending moment M is proportional to the maximum deflection a of the column (Fig. 9), or

$$M = Pa.$$

Thus the total compressive stress in the extreme fibers on the concave side is

$$p = \frac{P}{A} + \frac{Paz}{I},$$

or, substituting for I its value Ar^2, as in the preceding chapter, we finally get

$$p = \frac{P}{A}\left(1 + \frac{az}{r^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Consider two similar columns, similarly loaded, so that their elastic curves are geometrically similar. Then, both the deflection and the dimension z of each column are proportional to its length, so that

$$a = q_1 l,$$
$$z = q_2 l,$$

---

* For a proof of formula (10) see any standard work on mechanics of materials.

where q_1 and q_2 are empirical coefficients. Substituting these values in Eq. (11) and denoting the product q_1q_2 by q, we get

$$p = \frac{P}{A}[1 + q\,(l/r)^2],$$

or

$$\frac{P}{A} = \frac{p}{1 + q\,(l/r)^2}. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This is Rankine's formula, in which q is an empirical coefficient and p is the actual maximum compressive stress on the concave side of the column. For short prisms the second term in the denominator is small as compared with unity, so that the formula gradually changes into the ordinary formula, $P/A = p$, for uniform compression. On the other hand, for very long columns, unity in the denominator may be neglected as compared with the other term, and we get

$$\frac{P}{A} = \frac{1}{q} \cdot \frac{p}{(l/r)^2}. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

(4) *Ritter's Formula.* — In Eq. (12) the coefficient q may be so determined that for very long columns this formula would give the same value of buckling load as Euler's formula (1). In Euler's formula P/A is the stress in the material up to or beyond the elastic limit p_e, so that equating the right-hand members of Eqs. (1) and (13) under this supposition, we have

$$\frac{\pi^2 E}{(l/r)^2} = \frac{1}{q}\,\frac{p_e}{(l/r)^2}, \quad . \quad . \quad . \quad . \quad (14)$$

whence

$$q = \frac{p_e}{\pi^2 E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Formula (12) then becomes

$$\frac{P}{A} = \frac{p}{1 + \frac{p_e}{\pi^2 E}(l/r)^2}. \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

In this form it is known as Ritter's formula.

It must be clearly understood that Rankine's formula (12) may be used for determining either the safe load or the crushing load of a column, as long as q remains an *empirical* coefficient, determined directly from experiments. However, when q is ascribed a *theoretical* value, as in formula (16), the actual stress p cannot exceed the elastic limit of the material, as will be seen from the method of deducing the formula. Therefore, formula (16) cannot be made to check *directly* with results of experiments on failure of columns. However, the same objection may be raised against many *rational* formulæ used in engineering work without in the least impairing their usefulness for design.

(5) *Vierendeel's Formula.* — Prof. A. Vierendeel of the University of Louvain (Belgium) has shown that a formula similar to Rankine's may be deduced directly from Euler's formula, by considering the modulus of elasticity E to be a variable quantity. Namely, experiments show that beyond the elastic limit the coefficient of proportionality E between the stress and the strain decreases with the increase in the actual stress to which the material is subjected. For steel and iron a straight-line law may be assumed

$$E = a - b\frac{P}{A}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

where a and b are empirical coefficients. Vierendeel states that he is justified in using this empirical expression for E in formula (1) since Euler's formula is supposed to give a critical load, which strains the material almost to its ultimate strength. Thus, he obtains

$$\frac{P}{A}=\frac{\pi^2(a-bP/A)}{(l/r)^2},$$

or, solving for P/A,

$$\frac{P}{A}=\frac{\pi^2 a}{\pi^2 b+(l/r)^2}. \quad . \quad . \quad . \quad . \quad (18)$$

This formula is mathematically identical with formula (12), though the coefficients a and b have an entirely different physical meaning. Formula (18) is intended for columns of medium length, for which the critical unit stress P/A comes out beyond the elastic limit of the material. For long columns Euler's formula (1) must be used. The value of E in Eq. (1) and the values of a and b in Eq. (15) must be so selected that there is no discontinuity in the values of the critical load, and both formulæ must give the same result when P/A is equal to the elastic limit of the material.

Vierendeel's formula cannot, strictly speaking, be considered a rational formula, but it checks quite closely with the results of numerous experiments and for this reason ought to be useful in practice. An advantage of this formula over Rankine's is that it contains two empirical coefficients, instead of one, and therefore can be made to represent the results of experiments much more accurately.

Prob. 1. — The curve in Fig. 16 gives the results of certain experiments on the buckling of wrought-iron columns with pivoted ends. Show how closely Euler's curve covers the points corresponding to high values of l/r, and determine Johnson's straight line (both analytically and graphically) for low values of l/r. Take $E = 1.75$ million kg. per sq. cm., and $p = 3000$ kg. per sq. cm.

Prob. 2. — Prove that the ordinate of the point of tangency between Euler's curve and Johnson's straight line always corresponds to one-third of the ultimate strength p of the material.

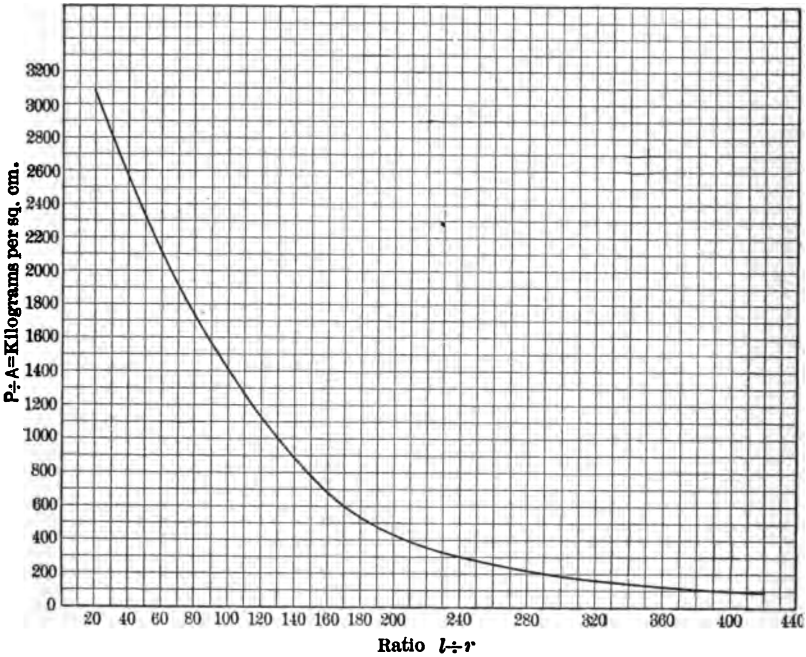

FIG. 16. — An experimental curve of critical load for certain wrought-iron columns with hinged ends.

Prob. 3. — Suppose that the straight line *ca* (Fig. 15) does not represent the results of certain experiments with sufficient accuracy, so that a more complicated curve is required to cover the range between *a* and *c*. Let this curve be required to pass through a given point *h* (x_0, y_0). Suggest a few simple possible equations for the curve *cha*, for instance $y = p - k_1x \mp k_2x^2$, or $y = \dfrac{a \pm mx}{b \pm nx}$, and show how to determine the values of the coefficients in these equations in order that the curve be tangent to Euler's curve.

Prob. 4. — Represent the experimental curve shown in Fig. 16 by an equation of the form

$$\frac{P}{A}=\frac{C}{(l/r)^m};$$

select the coefficients C and m so as to cover the widest possible range of values of (l/r). Try logarithmic cross section paper. Consult the article on the most probable curve in Part III (Thermodynamics), Chapter III.

Prob. 5. — For very long columns the critical load varies inversely as the square of the length of the column. Show by means of the experimental curve and of the formulæ given above that for columns of medium length the variations in strength are less than the square of the length, and that for very short prisms the critical load is independent of the length of the block.

Prob. 6. — Determine p and q in Rankine's formula so as to cover as well as possible the values of critical stress for the lower and medium values of l/r in Fig. 16.

Prob. 7. — Draw a curve of load per sq. cm. of cross section that would strain the columns in Fig. 16 up to the elastic limit, according to Ritter's formula. The elastic limit of the material is about one half of the crushing stress given in Prob. 1. Keep in mind that Ritter's formula does not give values of crushing loads, but of safe loads only.

Prob. 8. — Express the experimental results shown in Fig. 16 by means of Vierendeel's formula for medium values of l/r, and by means of Euler's formula for large values of l/r. Find the value of l/r beyond which Euler's formula must be used. Are the two curves tangent to each other, and if not, what is the angle between them?

Prob. 9. — The external diameter of a hollow circular column is 38 mm., the internal diameter is 25 mm., and the length is 25 m. The ends of the column are provided with flanges and are securely bolted to the floor and ceiling beams. What value of l/r should be used in calculating the strength of the column? Consult the preceding chapter in regard to the effective length of a column with the ends fastened.

CHAPTER VI.

STRENGTH OF THICK CYLINDERS AND SPHERES.

LET Fig. 17 represent a cross section of a steel cylinder with thick walls. Such cylinders are used for hydraulic presses, also as containers for compressed gases, and as barrels for big guns. Let the cylinder be subjected to a given internal pressure p_0 kg. per sq. cm. due to a gas or a liquid. The problem is to find the mechanical stresses in the wall, or to determine the thickness of the wall such that the maximum stress will not exceed a prescribed safe limit.

The stresses within the wall are tangential, that is, perpendicular to radii, and oppose the bursting of the cylinder. If the cylinder be thought of as consisting of concentric shells of infinitesimal thickness, the stress varies from shell to shell; the tangential tension is a maximum at the inner surface and reaches its minimum at the outer surface of the cylinder. The difficulty in the problem is to find the exact law of distribution of stress within the wall, so as to be able to calculate the stresses.

This problem of finding the distribution of stress in a thick cylinder is of interest to a student of applied mathematics, because he can follow in it a gradual development of the theory, and he can compare the improvements made in the fundamental assumptions. Five successive assumptions or approximations are made in the treatment given below, and the results so derived are analyzed. The first approximation is the simplest and the crudest, the last is in accordance with the modern views on the theory of elasticity, though it leads to much more complicated mathematics.

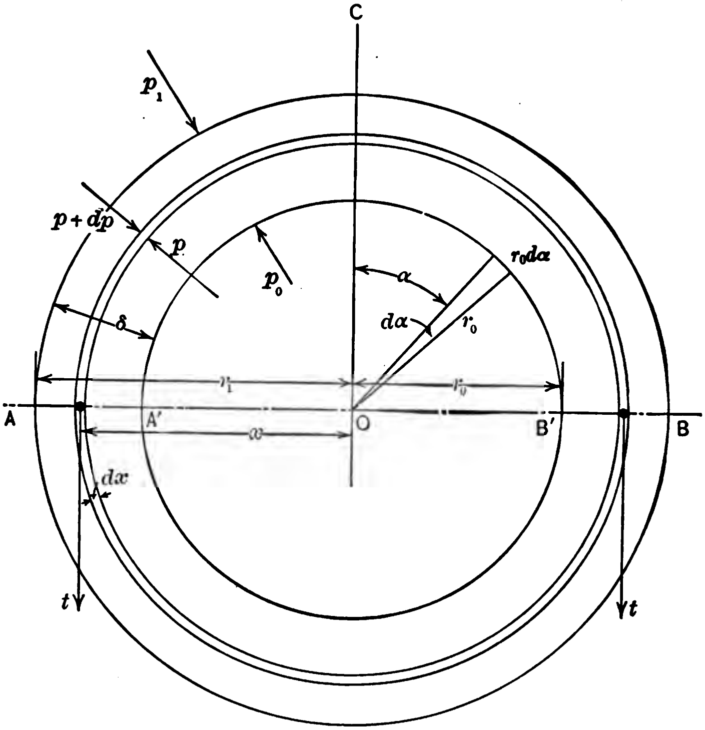

FIG. 17. — Cross section of a thick hollow cylinder or sphere.

The problem of the strength of a thick hollow sphere subjected to internal pressure is similar to that of the cylinder, and is briefly treated in parallel with it, in order to give the student an opportunity of applying the various assumptions made for the cylinder to a body of a different geometric shape.

Let the thickness of the wall (Fig. 17) be δ cm., the inner radius r_0 cm., the outer radius r_1 cm., and the internal pressure p_0 kg. per sq. cm. We shall assume at first that this internal pressure is so great that the external atmospheric pressure p_1 may be neglected in comparison.

Should the cylinder burst under the influence of this internal pressure, the two parts of it would separate along a plane such as *AB*, so the condition for equilibrium is that the sum of the upward vertical components of gas pressure on the upper half of the cylinder is equal to the sum of the vertical stresses along *AA′* and *BB′*.

Consider a slice of the cylinder one cm. long in the direction perpendicular to the plane of the paper. The surface corresponding to an infinitesimal central angle $d\alpha$ is $r_0\,d\alpha$, and the normal pressure is $p_0r_0\,d\alpha$. But for each element such as $r_0\,d\alpha$ there is a symmetrical element on the other side of the plane *OC*, so that the horizontal components of the pressure are balanced and the vertical components are added. The vertical component of the pressure upon either element is

$$p_0r_0\,d\alpha\cos\alpha,$$

so that the total pressure on either quadrant is

$$\int_0^{\frac{1}{2}\pi} p_0r_0\cos\alpha\,d\alpha = p_0r_0.$$

Since $r_0 \times 1$ is the area of the plane *A′O*, the foregoing result may be interpreted by saying that the total vertical pressure is the same as if it were distributed uniformly upon the diametral plane of the cylinder.

Let the variable unit stress along *AA′* be denoted by *t*, and the distance from the axis by *x*. Then the infinitesimal force acting upon an element of the plane

AA', per unit of length in the direction perpendicular to the plane of the paper, is $t\,dx$. Equating the total internal force resisting a rupture along AA' to the total pressure of gas, we get

$$\int_{r_0}^{r_1} t\,dx = p_0 r_0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the fundamental equation for the cylinder. In order to be able to integrate the left-hand side it is necessary to make an assumption regarding the distribution of t as a function of the distance x. This is done in the various approximations that follow.

Let now Fig. 17 represent the cross section of a thick sphere; the area of the diametral plane $A'B'$ is πr_0^2, and the upward pressure of the gas is $p_0 \pi r_0^2$. The stress corresponding and perpendicular to an infinitesimal strip of plane AA' of width dx is $t \cdot 2\,\pi x \cdot dx$. Thus, equating the pressure of the gas to the internal stress produced by it we have

$$\int_{r_0}^{r_1} t \cdot 2\,\pi x \cdot dx = p_0 \pi r_0^2,$$

or, dividing by $2\,\pi$,

$$\int_{r_0}^{r_1} t\,x\,dx = \tfrac{1}{2}\,p_0 r_0^2. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is the fundamental equation for the sphere. A reasonable assumption must be made concerning t as a function of x, before this equation can be integrated. Some of such assumptions are discussed below.

First Approximation.

The Stress is Distributed Uniformly.

(*a*) *Cylinder.* — The simplest assumption that can be made in Eq. (1) is

$$t = t_0 = \text{const.}; \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in other words, the stress is distributed uniformly through the thickness of the wall. This assumption is practically correct for thin walls, for instance, in steam boilers, and it may be accurate enough for comparatively thick walls, where particular accuracy is not required.

Substituting a constant value of t_0 in Eq. (1), and integrating, we get

$$t_0 (r_1 - r_0) = p_0 r_0,$$

or

$$t_0 \delta = p_0 r_0, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where δ is the thickness of the wall. Any one of the four quantities which enter this formula may be determined if the remaining three are known.

If the external pressure p_1 cannot be neglected, Eq. (4) becomes

$$t_0 \delta = p_0 r_0 - p_1 r_1. \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

(*b*) *Sphere.* — Using condition (3) in Eq. (2) we get

$$t_0 (r_1^2 - r_0^2) = p_0 r_0^2,$$

or

$$t_0 \delta (r_1 + r_0) = p_0 r_0^2. \quad . \quad . \quad . \quad . \quad (5)$$

For thin walls r_1 is approximately equal to r_0, and the formula is simplified to

$$2\, t_0 \delta = p_0 r_0. \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

Prob. 1. Prove Eq. (4a) and explain the reason why δ, according to this formula, assumes a negative value when the external and the internal pressures are equal.

Prob. 2. Instead of bursting along a diametral plane, a cylinder may burst along a cross section perpendicular to its longitudinal axis, due to pressures on the enclosed ends. Show that under the assumption of a uniform distribution of stress and thin walls, a cylinder possesses a double factor of safety against such a rupture as compared to a failure along a diametral plane. *Hint.* — Let the unit stress parallel to the longitudinal axis be q kg. per sq. cm.; the resisting stress is approximately equal to $2\pi r_0 \delta q$. The pressure of the gas on the ends is $\pi r_0^2 p_0$.

Prob. 3. Apply Eqs. (5) and (5a) to the case in which the external pressure p_1 cannot be neglected.

Second Approximation.

The Thickness of the Wall is Assumed Unchanged Under Stress.

(*a*) *Cylinder.* — A more correct idea of the distribution of the stress t throughout the cylinder wall is obtained by analyzing critically the foregoing assumption of t = constant. According to the fundamental law of the theory of elasticity, stresses are proportional to deformations which they produce (Hooke's law). Let l be the original length of a fiber subjected to a unit stress t, and let Δl be the elongation of this fiber under the influence of t. Then the *relative* elongation (or the elongation per unit length) is $\Delta l/l$, and the above law is expressed mathematically thus:

$$t = E\frac{\Delta l}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where E is the so-called coefficient (or modulus) of elasticity of the material. This law holds true as long as the elastic limit of the material is not exceeded. Applying this law with the assumption that t is constant, it is shown below that one gets the absurd result that under the influence of an internal pressure various concentric annuli of the cylinder tend to spread apart. Common sense would indicate that an internal pressure tends to press the inner annuli or layers against the outer ones, because only in this way can the outer layers be made to help in the resistance to bursting.

Let u be the increase in the radius x of an infinitesimal shell within the wall (Fig. 17), under the influence of the internal pressure; in other words, the particles of the material which originally are at a distance x from the center of the cylinder occupy a circle of radius $x + u$. According to the above mentioned law of elasticity, the tangential stress to which these particles are subjected is

$$t = E \cdot \frac{2\pi (x + u) - 2\pi x}{2\pi x}, \quad . \quad . \quad . \quad (7)$$

or

$$t = E\frac{u}{x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7a)$$

If t be assumed constant, as in the first approximation, then u must be proportional to x; this means that the external annuli expand more than the internal ones, all the annuli or layers tending to spread apart, which is evidently absurd. In reality u decreases with the increasing x, so that the internal annuli press upon the external ones. Not knowing the actual law relating u and x, the next best assumption is that u is constant,

$$u = c; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in other words, *there is no spreading or compression between the concentric layers and the total thickness of the wall remains unchanged by the pressure.* Eq. (7a) then indicates that t varies inversely with x, so that the inner parts of the wall are subjected to greater stresses than the outer parts. The results thus obtained, while not altogether correct, must be more accurate than the relationship stated in Eq. (4).

Substituting the value of t from Eq. (7a) into Eq. (1) and integrating gives

$$Eu \operatorname{Ln}\frac{r_1}{r_0} = p_0 r_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Eliminating the product uE from Eqs. (7a) and (9) we obtain the following formula for t:

$$t = \frac{p_0 r_0}{x \operatorname{Ln} (r_1/r_0)}. \quad . \quad . \quad . \quad . \quad (10)$$

The maximum tensile stress is at the inner surface of the cylinder where $x = r_0$, and is equal to

$$t_{\max} = \frac{p_0}{\operatorname{Ln} (r_1/r_0)}. \quad . \quad . \quad . \quad . \quad (11)$$

(*b*) *Sphere.* — It is assumed in Eq. (7) that all the stresses are in planes perpendicular to the longitudinal axis of the cylinder, so that the value of the stress depends upon the *linear expansion* of the wall. In the case of a sphere the stresses at a point have the same value in all directions in a plane perpendicular to the radius. Each infinitesimal concentric spherical shell remains a sphere after expansion, so that stresses may be said to depend upon the *surface expansion.* By analogy with Eq. (7), we may write

$$t = E\frac{4\,\pi\,(x+u)^2 - 4\,\pi x^2}{4\,\pi x^2}, \quad . \quad . \quad . \quad . \quad (12)$$

where $4\,\pi x^2$ is the area of the infinitesimal shell within the wall. This equation is simplified to

$$t = E\frac{2\,ux + u^2}{x^2}.$$

We consider here deformations u which are extremely small as compared with the radius x of the shell. Therefore u^2 in the numerator of the preceding formula may be neglected without any appreciable error, and we get

$$t = E\frac{2\,u}{x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12a)$$

Eq. (12a) may also be written directly from Eq. (7), by considering a small square piece in the infinitesimal shell under consideration. This element is subjected to stresses t on all four sides adjacent to the rest of the shell, and each pair of equal and opposite forces t not only stretches the element, but also counterbalances the contracting action due to the other pair of forces t. Assume that it is necessary to apply a tension stress equal to $\frac{1}{2}t$ to counterbalance the secondary compression effect due to a tension stress t in a perpendicular direction. The conclusion is that all the stresses t must be doubled for a given u in a spherical shell, as compared with a cylindrical shell. It will be seen accordingly that t has a double value in Eq. (12a) as compared with Eq. (7).*

Proceeding as in the case of the cylinder we obtain

$$t = \frac{p_0 r_0^2}{2x(r_1 - r_0)}. \quad . \quad . \quad . \quad . \quad (13)$$

Prob. 4. — The internal diameter of a steel cylinder is 20 cm.; the wall is 10 cm. thick; the maximum safe permissible stress t in the material is 1200 kg. per sq. cm. Use this cylinder to make the foregoing theory clear to yourself. For example, (*a*) determine the permissible gas pressure when the first and the second approximations of the theory are used respectively; (*b*) assuming the cylinder to be just safe according to the first approximation, determine the external diameter and the extra weight of steel, in per cent, necessary to make the wall sufficiently thick to be safe according to the second approximation; (*c*) make a drawing of the cross section of the cylinder to scale, and show graphically, by means of a curve, the distribution of stresses in the wall, according to both approximations. Perform all computations very carefully, and make the sketch to a large scale, because the results are later to be supplemented with those obtained according to more accurate approximations.

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* When the student comes to the fourth approximation he will see that the effect of the lateral compression is not taken quite correctly into account in Eq. (12a). However, for a thin sphere the result (13), based upon Eq. (12a), becomes identical with Eq. (5a). This fact shows the reasonableness of assumption (12), at least as an approximation.

Prob. 5. — Plot a curve giving the ratio of the external to the internal diameter of forged steel guns for explosion pressures ranging from 3000 to 7000 kg. per sq. cm. The tensile stress in the material is allowed to go almost to the elastic limit, which in this case is approximately 3500 kg. per sq. cm. Draw the curve to a large scale because other curves are to be drawn according to more accurate approximations on the same sheet.

Prob. 6. — Check the integration which leads to Eq. (13).

Prob. 7. — Solve Prob. 4 for a sphere, using the same data.

Prob. 8. — Modify formula (10) for the case in which the external pressure cannot be neglected.

Prob. 9. — Modify formula (13) for the case when the external pressure cannot be neglected.

Prob. 10. — Prove that for a very thin spherical shell, formula (13) becomes identical with Eq. (5a).

Third Approximation.

The Volume of the Wall is Assumed to Remain Unchanged under Stress.

One obvious criticism of assumption (8) is that it indicates an increase in the volume of the walls when a cylinder is subjected to an internal pressure. The original volume of the wall of a cylinder of length l is

$$V = \pi l \,(r_1^2 - r_0^2),$$

while the volume of the same cylinder under pressure, according to the second approximation, is

$$V' = \pi l \,[(r_1 + u)^2 - (r_0 + u)^2].$$

The difference

$$V' - V = \pi l \cdot 2\,u\,(r_1 - r_0)$$

is positive, so that there would be an increase in volume if u were constant. It is much more reasonable, however, to believe that the total volume of the wall decreases under pressure, the inner layers being pressed against the outer ones. The actual decrease in volume not being known, the next best thing to assume is that there is no increase in volume. This is equivalent to saying that the tension t is distributed still less uniformly than in the second approximation, the values of u being larger at the inner layers and smaller at the external layers, so that the inside annuli press upon the outside ones.

(*a*) *Cylinder.* — Let the values of u at the inside and the outside surfaces be u_0 and u_1 respectively. Then the condition that there is no increase in volume becomes

$$V' - V = \pi l\,[(r_1 + u_1)^2 - (r_0 + u_0)^2] - \pi l\,(r_1^2 - r_0^2) = 0,$$

or, after simplification,

$$2\,r_1u_1 + u_1^2 = 2\,r_0u_0 + u_0^2.$$

The elongations u_1 and u_0 being very small compared with the radii of the cylinder, their squares may safely be neglected, and we finally get

$$r_1u_1 = r_0u_0.$$

This means that the extreme elongations are inversely as the radii of the cylinder. It is reasonable to assume (though this is not logically necessary) that the same law holds inside the wall, or

$$r_1u_1 = xu = r_0u_0 = \text{Const.*} \quad . \quad . \quad . \quad (14)$$

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* Any function of the form

$$u = \frac{\text{Const.}}{x} + (x - r_0)\,(r_1 - x)\,f\,(x)$$

satisfies condition (14) for $x = r_1$ and $x = r_0$, provided that $f\,(x)$ does not become infinite at these values. We merely make the simplest assumption that $f\,(x) = 0$.

Substituting the value of u from this expression into the general formula (7a) for t, we get

$$t = E\frac{r_0 u_0}{x^2}, \quad . \quad . \quad . \quad . \quad (14a)$$

and Eq. (1) gives

$$Eu_0\left(\frac{1}{r_0} - \frac{1}{r_1}\right) = p_0.$$

Eliminating the product Eu_0 from this formula by means of Eq. (14a) we finally get

$$t = p_0\left(\frac{r_0}{x}\right)^2 \cdot \frac{r_1}{r_1 - r_0}. \quad . \quad . \quad . \quad . \quad (15)$$

The maximum tension is at the inner surface, and is equal to

$$t_{\max} = p_0\frac{r_1}{r_1 - r_0}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

From this expression the necessary thickness of the wall for a given pressure and with a given inner diameter may be calculated.

(*b*) *Sphere.* — The condition for no increase in volume is

$$V' - V = \tfrac{4}{3}\pi\left[(r_1 + u_1)^3 - (r_0 + u_0)^3\right] - \tfrac{4}{3}\pi(r_1^3 - r_0^3) = 0,$$

or, neglecting again the higher powers of u_1 and u_0,

$$r_1^2 u_1 = r_0^2 u_0.$$

Assuming, as in the case of the cylinder, that the same law holds good within the wall, we obtain the relation

$$r_1^2 u_1 = x^2 u = r_0^2 u_0 = \text{Const.} \quad . \quad . \quad . \quad (17)$$

Hence, Eq. (12a) becomes

$$t = E\frac{2\,r_0^2 u_0}{x^3}, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and Eq. (2) becomes, after integration,

$$E \cdot 2\, r_0^2 u_0 \left(\frac{1}{r_0} - \frac{1}{r_1}\right) = \frac{1}{2} p_0 r_0^2. \quad . \quad . \quad (19)$$

Substituting the value of Eu_0 from Eq. (18) into Eq. (19) we finally get

$$t = \frac{1}{2} p_0 \left(\frac{r_0}{x}\right)^3 \cdot \frac{r_1}{r_1 - r_0}, \quad . \quad . \quad . \quad . \quad (20)$$

a formula similar to Eq. (15) for the cylinder.

Prob. 11. — Continue Prob. 4, using the third approximation; answer questions (*a*), (*b*), and (*c*). Supplement the drawing by showing the external diameter of the cylinder required, according to the three approximations, for the same gas pressure. Plot a curve of distribution of u knowing that for steel the modulus of elasticity E is equal to about 2 million kg. per sq. cm.

Prob. 12. — In the first approximation t is independent of x, in the second approximation t is inversely proportional to x, in the third approximation t is inversely proportional to x^2. Assuming that there are physical reasons why t should vary inversely as the nth power of x, n being a known number, write an expression for $t_{\max}$ similar to Eq. (16).

Prob. 13. — For some materials (for instance, for cast iron), elongations $\Delta l/l$ in formula (6) are not proportional to tensions t, but increase more rapidly; in other words, the material is relatively weaker for greater stresses. Assume that the relation between the tension and the elongation may be represented by an empirical formula

$$t = E'\left(\frac{\Delta l}{l}\right) - E''\left(\frac{\Delta l}{l}\right)^2,$$

where E' and E'' are given constants, and deduce an expression for $t_{\max}$. Prove that with such a material, tensions are distributed more uniformly than with one that has a constant modulus of elasticity E.

Prob. 14. — Modify formulas (15) and (20) for the case in which the external pressure cannot be neglected.

Prob. 15. — Continue Prob. 7 for the sphere, according to the requirements stated in Prob. 11.

Prob. 16. — A set of spherical steel shells are to be cast for calorimetric tests of certain explosives. Show that these shells can be designed according to the formula

$$\frac{\delta}{D_0} = \frac{p_0}{4\,t_{\max} - 2\,p_0},$$

where δ is the thickness of the shell, D_0 the inside diameter, and $t_{\max}$ is the allowable working stress in the material.

Prob. 17. — Referring to Prob. 5, plot on the same sheet a similar curve according to the third approximation.

Fourth Approximation.

Lateral Deformations Taken into Account.

(*a*) *Cylinder.* — An infinitesimal shell of radius x and thickness dx (Fig. 17) is subjected to three kinds of stresses, at right angles to each other:

(1) Tangential tension t;
(2) Radial compression p;
(3) Tension q parallel to the axis of the cylinder.

The tangential tension resists the bursting of the cylinder and the radial compression is due to the inner layers of the wall pressing upon the outer layers under the influence of the gas contained in the cylinder. The longitudinal tension q is caused by the gas pressure on the enclosed ends of the cylinder; this is a stress which has heretofore been neglected. At a reasonable distance from the ends, stress q may be considered to be uniformly distributed over the cross section of the wall, so that

$$q = q_{\text{ave}} = \frac{p_0 r_0^2 - p_1 r_1^2}{r_1^2 - r_0^2}. \quad . \quad . \quad . \quad (21)$$

Two more equations are necessary in order to determine t and p as functions of the radius x. One equation is obtained by considering the infinitesimal shell of thickness dx as a thin cylinder subjected to pressure p on the inside, and to opposing pressure $p + dp$ on the outside. Eq. (4) applies to this case exactly, because t may be considered constant over the infinitesimal thickness dx. We thus have

$$t\,dx = px - (p + dp)(x + dx),$$

or, neglecting the infinitesimal quantity $dp\,dx$ of the second order,

$$t\,dx = -d\,(px). \quad . \quad . \quad . \quad . \quad . \quad (22)$$

A third relation between p, q, and t is obtained from the so-called law of lateral deformations. Namely, experiment shows that when an elastic body, say a rod, is subjected to a longitudinal tension t, it not only expands in the direction of t, but it also contracts in planes perpendicular to t. This contraction, not caused directly by external forces, is called lateral contraction. Similarly, if the rod were compressed by an external force, the lateral deformation would be an expansion. Experiment further shows that lateral deformations are proportional to direct deformations, so that if a stress t produces an extension $\Delta l/l$, the lateral contraction is equal to

$$\frac{1}{m}\left(\frac{\Delta l}{l}\right),$$

where m is a numerical coefficient. In this country, this coefficient is usually assumed to be equal to 3. In France, where this coefficient is known as Poisson's ratio, it is taken equal to 4; C. Bach in Germany uses $m = 10/3$. Another way of stating the preceding law is that when a stress t is produced in a certain direction, stresses equal to $-t/m$ are produced at right angles to t.

Thus, applying this law to our case, we can say that the longitudinal expansion of the cylinder in the direction of its axis is an algebraic result (*a*) of the direct tension produced by the pressures on its ends; (*b*) of the lateral contraction caused by tension t, and (*c*) of the lateral expansion produced by compression p. In other words, the annulus elongates in the direction of the axis of the cylinder as if an effective unit tension q_{eff} were applied to it, where

$$q_{\text{eff}} = q - \frac{t}{m} + \frac{p}{m}. \quad . \quad . \quad . \quad . \quad (23)$$

In this equation we shall now make a simplifying assumption, namely, that the plane cross sections of the cylinder remain planes after deformation. This condition is fulfilled in practice, since the ends of the cylinder are sufficiently stiff to withstand any appreciable change in form. In other words, q_{eff} is independent of x, being the same for all annuli. But, as has been stated above, the applied longitudinal stress, q, is also independent of x, at least at a reasonable distance from the ends of the cylinder. Hence, according to Eq. (23), $(t - p)$ must also be independent of x, or

$$t - p = 2A = \text{Const.}, \quad . \quad . \quad . \quad (24)$$

where $2A$ is a constant to be determined later.

From Eqs. (21), (22), and (24) the three unknown stresses p, q, and t may be calculated. Of these, q is directly known from Eq. (21); so that Eqs. (22) and (24) may be solved together for p and t. Substituting the value of t from Eq. (24) into Eq. (22) gives

$$(p + 2A)\,dx = -(p\,dx + x\,dp),$$

or

$$-2A\,dx = 2p\,dx + x\,dp.$$

This differential equation is easily integrated by multiplying both sides by x; it is reduced then to

$$-2\,Ax\,dx = d\,(x^2 p),$$

or, after integration,

$$-Ax^2 + B = x^2 p,$$

where B is the constant of integration. Solving for p gives

$$p = \frac{B}{x^2} - A. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

Hence, from Eq. (24),

$$t = p + 2\,A = \frac{B}{x^2} + A. \quad . \quad . \quad . \qquad (26)$$

The constants A and B are determined from Eq. (25), knowing that $p = p_0$ inside the cylinder and $p = p_1$ outside the cylinder, so that

$$\left.\begin{aligned} \frac{B}{r_0^2} - A &= p_0, \\ \frac{B}{r_1^2} - A &= p_1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

Solving these equations for A and B, it will be found that both A and B are positive. Hence, a comparison of Eqs. (25) and (26) will easily show that the tangential tension t is always numerically greater than the radial compression p. Both reach their maximum for $x = r_0$, viz., at the inner surface of the cylinder.

(*b*) *Sphere.* — Consider again the infinitesimal shell of diameter x and of thickness dx. The shell is in equilibrium under the influence of pressure p on its inner surface, pressure $(p + dp)$ on the outside surface, and tension t. Applying Eq. (5) to this case, we get

$$2\,tx\,dx = px^2 - (p + dp)\,(x + dx)^2,$$

or, performing the multiplication and neglecting the infinitesimal terms of higher order,

$$2\,tx\,dx = -d\,(px^2). \quad . \quad . \quad . \quad . \quad (28)$$

This equation takes the place of Eq. (22) for the cylinder. The law of lateral deformations furnishes two equations for the sphere:

$$t_{\text{eff}} = E\frac{u}{x} = t - \frac{t}{m} + \frac{p}{m}; \quad . \quad . \quad . \quad (29)$$

$$p_{\text{eff}} = -E\frac{du}{dx} = p + \frac{t}{m} + \frac{t}{m}. \quad . \quad . \quad (30)$$

To deduce Eq. (29) consider a small square portion of the infinitesimal shell of thickness dx. This portion expands in two perpendicular directions under the influence of the tensions t, and is compressed radially by the pressure p. Thus, according to the law of lateral deformations,

$$t_{\text{eff}} = t - \frac{t}{m} + \frac{p}{m}.$$

But, according to the fundamental Eq. (7a),

$$t_{\text{eff}} = E\frac{u}{x},$$

from which Eq. (29) follows. By comparing Eq. (29) with Eq. (12a) it will be seen that if p could be neglected and if m were equal to 2, the two equations would be identical. This shows the error, or rather the assumptions, made in Eq. (12a). It is proper to use Eq. (7a) for the sphere, because it is applied here to the effective tension, the lateral deformations having been properly accounted for. Eq. (30) expresses the fact that the radial pressure p is further increased by the lateral pressures t/m due to the tangential tensions t in two mutually perpendicular directions. Thus the effective radial pressure is

$$p_{\text{eff}} = p + \frac{t}{m} + \frac{t}{m}.$$

On the other hand, it is proved below that du/dx represents the relative radial compression which in the case of dilatation is denoted by $\Delta l/l$ in Eq. (6). Hence,

$$p_{\text{eff}} = -E\frac{du}{dx},$$

and Eq. (30) follows immediately.

To prove that du/dx represents the relative contraction, consider a point a on the inside surface of the infinitesimal shell, and a point b, on the same radius, but on the outside surface of the same shell. The original distance between the points is dx. After the deformation, the distance of a from the center becomes $x + u$, while that of b becomes $(x + dx) + (u + du)$. Now

$$\frac{\Delta l}{l} = \frac{\text{new length } ab - \text{original length } ab}{\text{original length } ab},$$

or

$$\frac{\Delta l}{l} = \frac{\{[(x + dx) + (u + du)] - (x + u)\} - dx}{dx} = \frac{du}{dx},$$

which was to be proved. The minus sign in Eq. (30) is necessary before du/dx, because u decreases from the center to the periphery and consequently du/dx is negative, while the right-hand side of the equation is essentially positive.

The three equations (28), (29), and (30) contain three unknown functions of x, namely p, t, and u. To solve these equations we substitute the values of t

and p from Eqs. (29) and (30) into Eq. (28). This gives a differential equation between u and x, so that u can be determined as a function of x. Substituting this value of u in Eqs. (29) and (30) p and t are expressed as functions of x, and the problem is thus solved.

Solving (29) and (30) for p and t gives

$$t = \frac{mE}{m^2 - m - 2}\left(\frac{du}{dx} + m\frac{u}{x}\right), \quad . \quad . \quad . \quad (31)$$

$$-p = \frac{mE}{m^2 - m - 2}\left((m-1)\frac{du}{dx} + \frac{2u}{x}\right). \quad (32)$$

Substituting these values in Eq. (28) we obtain, after reduction,

$$x^2\frac{d^2u}{dx^2} + 2x\frac{du}{dx} = 2u. \quad . \quad . \quad . \quad . \quad (33)$$

This equation may be integrated according to the general rules for the integration of differential equations of this type,* but it can also be solved as follows: Multiply both sides by x and add to both sides $x^2\,du/dx$; the result is

$$x^3\frac{d^2u}{dx^2} + 3x^2\frac{du}{dx} = 2ux + x^2\frac{du}{dx},$$

or

$$\frac{d}{dx}\left(x^3\frac{du}{dx}\right) = \frac{d}{dx}(x^2u).$$

Integrating both sides, we get

$$x^3\frac{du}{dx} = x^2u - 3C_2, \quad . \quad . \quad . \quad . \quad (34)$$

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* See Probs. 25 and 26 below.

where $3\,C_2$ is a constant of integration. To integrate again, we divide both sides of the equation by x^4; the result is

$$\frac{x \cdot \dfrac{du}{dx} - u}{x^2} = -\frac{3\,C_2}{x^4},$$

or

$$\frac{d}{dx}\left(\frac{u}{x}\right) = -\frac{3\,C_2}{x^4}.$$

Integrating both sides, we find

$$\frac{u}{x} = C_1 + \frac{C_2}{x^3},$$

where C_1 is another constant of integration. Solving for u gives

$$u = C_1 x + \frac{C_2}{x^2}. \quad . \quad . \quad . \quad . \quad . \quad (35)$$

This is the expression for u as a function of x. Substituting it in Eqs. (31) and (32) we find

$$t = E\left\{\frac{m}{m+1}\cdot\frac{C_2}{x^3} + \frac{m}{m-2}\cdot C_1\right\}, \quad . \quad . \quad (36)$$

$$p = E\left\{\frac{2m}{m+1}\cdot\frac{C_2}{x^3} - \frac{m}{m-2}\cdot C_1\right\}. \quad . \quad . \quad (37)$$

The constants C_2 and C_1 are determined by applying Eq. (37) to the terminal conditions, viz., $p = p_0$ when $x = r_0$, and $p = p_1$ when $x = r_1$.

Prob. 18. — Continue Probs. 4 and 11, using the formulæ of the fourth approximation.

Prob. 19. — Prove that $A = q$, so that q_{eff} is never greater than from $\frac{1}{2}$ to $\frac{1}{3}\,q$, if $m = 3$ to 4.

Prob. 20. — Continue Probs. 5 and 17, using the formulæ of the fourth approximation.

Prob. 21. — From Eqs. (36) and (37) it follows that $2t - p =$ Const. This corresponds to Eq. (24) for the cylinder. Assuming this relation to be given, what is the simplest way to deduce Eqs. (36) and (37)?

Prob. 22. — Represent Eqs. (36) and (37) in their final practical form, without C_1 and C_2.

Prob. 23. — Show that the formula given in Prob. 16 should be modified to

$$\frac{\delta}{D_0} = \frac{1}{2}\left(\sqrt[3]{\frac{t_{\max} + p_0}{t_{\max} - p_0}} - 1\right).$$

Prob. 24. — Continue Probs. 7 and 15, using the formulæ of the fourth approximation.

Prob. 25. — Solve Eq. (33) by first reducing it to one with constant coefficients, and then solving the latter by the usual method of the auxiliary equation. *Hint.* — Use the substitution $x = e^s$.

Prob. 26. — Solve Eq. (33) by assuming *a priori* that the solution is of the form

$$u = C_1 x^{n_1} + C_2 x^{n_2}.$$

Fifth Approximation.

True Deformation as a Measure of Effective Stress.

(*a*) *Cylinder.* — Eqs. (21), (25), and (26) give the values of the component stresses at any point of the cylinder wall. Under the influence of these stresses the material is strained, each component stress producing a direct deformation in its own direction and lateral deformations of the opposite sign in perpendicular planes. Therefore, the resultant actual deformations are not proportional to the component stresses, but must be calculated from the component stresses according to the law of lateral deformations. It is not the stress, but the actual deformation that determines the safety of the material at a given point.

Let, for instance, a cube whose edge is one cm., be subjected to a compression of 100 kg. upon two opposite sides, and to a tension of 60 kg. upon two other opposite sides; let the third pair of sides have no forces applied to them. The total deformation in the direction normal to the first pair of sides corresponds not to a stress of 100 kg. per sq. cm., but to a stress of $100 + \frac{1}{4} \times 60 = 115$ kg. compression per sq. cm., assuming $m = 4$. Similarly, the deformation normal to the second pair of sides corresponds to a stress of $60 + \frac{1}{4} \times 100 = 85$ kg. per sq. cm., while in the direction perpendicular to the third pair of sides on which no external forces are applied, the deformation corresponds to a tensile stress of $\frac{1}{4} \times 100 - \frac{1}{4} \times 60 = 10$ kg. per sq. cm.

Thus, while expressions (21), (25), and (26) are correct in themselves, the values of p, q, and t calculated from these equations do not give the true stresses in the material. The true or the effective stress parallel to the axis of the cylinder is not q, but is q_{eff} as given by Eq. (23). Similarly, the true tangential stress is not t, but

$$t_{\text{eff}} = t - \frac{p}{m} + \frac{q}{m}, \qquad . \quad . \quad . \quad . \quad (38)$$

and the true radial compression is

$$p_{\text{eff}} = p + \frac{t}{m} + \frac{q}{m}. \qquad . \quad . \quad . \quad . \quad (39)$$

The effective values may be smaller or greater than the component stresses p and t, according to the values of the radii of the cylinder and of the applied pressures. Substituting the values of t and p from Eqs. (25) and (26) into Eqs. (23), (38), and (39), and remembering that $A = q$ (see Prob. 19), we get

$$\left.\begin{aligned} q_{\text{eff}} &= q\left(1-\frac{2}{m}\right); \\ p_{\text{eff}} &= \left(1+\frac{1}{m}\right)\frac{B}{x^2}-q\left(1-\frac{2}{m}\right); \\ t_{\text{eff}} &= \left(1+\frac{1}{m}\right)\frac{B}{x^2}+q\left(1-\frac{2}{m}\right). \end{aligned}\right\} \quad . \quad . \quad . \quad (40)$$

The value of t_{eff} at the inner surface of the cylinder usually determines the required thickness for a given factor of safety, or vice versa.

(*b*) *Sphere.* — Instead of being given by Eq. (36), the true tangential stress, according to Eqs. (29) and (35), is equal to

$$t_{\text{eff}} = E\left(\frac{C_2}{x^3}+C_1\right). \quad . \quad . \quad . \quad . \quad (41)$$

Similarly

$$p_{\text{eff}} = E\left(\frac{2\,C_2}{x^3}-C_1\right). \quad . \quad . \quad . \quad (42)$$

The constants C_1 and C_2 are determined, as before, by applying Eq. (37) to the terminal conditions.

Prob. 27. — Continue Probs. 4, 11, and 18, using the formulæ of the fifth approximation.

Prob. 28. — Write down formulæ for the true stresses in the material of a cylinder with very thin walls.

Prob. 29. — Deduce formulæ for the strength of a thick cylinder subjected to an external pressure, for example, a fire tube in a steam boiler, or a pipe culvert.

Prob. 30. — A compound gun is to be formed out of two concentric cylinders in the following way: The inside radius of the inner cylinder is to be r_0, the outside radius ρ. The inside radius of the outer cylinder is left less than ρ by a small amount α; the outside radius is to be r_1. The outer cylinder (sometimes called the jacket or the hoop) is heated so that its inside radius becomes equal to ρ, and then it is shrunk on the inner cylinder. Assuming r_0, ρ, and r_1 to be known, show how to determine α so that during firing when a pressure p_0 is developed inside, the effective tensile stresses on the inner surfaces of both cylinders will reach the elastic limit of the material.

Note. — The purpose of using hoops on guns is to make them lighter. In an ordinary gun the material is worked up to the safe limit only on the inner surface of the cylinder. The hoop gives an original compression to the inner cylinder, so that the explosion of the gases must first overcome this compression before it produces a tensile stress t. Again, the inner surface of the hoop may be worked up to the elastic limit, so that the material is utilized better than in a solid gun.

Prob. 31. — Determine the necessary thickness of the barrel of a gun whose internal diameter is 15 cm., and in which the pressure during firing reaches 4000 kg. per sq. cm. Then design the same gun as a compound cylinder with a hoop, selecting the radii of the internal cylinder and of the hoop so as to use a minimum of material. See Prob. 5 for the permissible stress.

Prob. 32. — Continue Probs. 5, 17, and 20, using the formulæ of the fifth approximation.

Prob. 33. — Continue Probs. 7, 15, and 24, using the formulæ of the fifth approximation.

Prob. 34. — Write down formulæ for true stresses and for deformation u for a very thin elastic sphere.

Prob. 35. — Write down formulæ (41) and (42) in their final practical form, with the constants C_1 and C_2 expressed through the known quantities.

Prob. 36. — The above given theory of thick cylinders presupposes that longitudinal tensions q are independent of x. This does not hold true near the ends of the cylinder. If a cylinder has, say, hemispherical ends, the stresses in the connecting plane come out different according to whether they are calculated from the cylindrical or the hemispherical side. It would be a suitable problem for original research to develop formulæ for stresses in that part where the cylinder connects with the hemisphere, so that the changes in the values of the stresses would come out to be gradual, as they are in reality, and not sudden.

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